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Collective Oscillations in a Simple Metal. I. Spin Waves*

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Using the Landau theory of Fermi liquids, we analyze the paramagnetic behavior of the alkali metals at low temperatures. An integral formulation of the kinetic equations is derived, and this is solved for arbitrary wavelengths (within the limits of Landau's theory), for arbitrary direction of propagation and retaining an arbitrarily large number of the Landau interaction coefficients. We derive a practical algorithm for obtaining the transverse and longitudinal susceptibilities; this enables us to discuss, with the generality just outlined, all of the collective magnetization modes, as well as the features of individual-particle excitations in a simple metal with spherical Fermi surface. The effects of scattering from dilute, random impurities are also included. We catalog systematically the dispersion properties as well as the oscillator strengths of all of these waves. Comparison of the results with the experiments of Schultz and Dunifer is described, and the feasibility of observing the various spin-wave modes is discussed.

I. INTRODUCTION

The value of Landau's[1,2] theory of Fermi liquids has, in the past few years, become widely appreciated. This recognition is particularly evident in the case of charged systems, where Silin[3,4] extended Landau's work. The best examples of such charged Fermi liquids are those formed by the conduction electrons in the alkali metals. Regarded in this sense the metals may be termed "simple."

The valuable experimental work of Schultz and Dunifer[5,6] and Walsh,[7] indicating the validity of the theory, has triggered the appearance of a host of papers[8–14] pointing out the richness of its detailed predictions about waves in these "simple" metals. For the most part these treatments have been directed towards the analysis of waves of weak spatial dispersion propagating in directions of high symmetry (with respect to a constant external magnetic field) and the effects of collisions usually ignored. For the purposes of comparison with experiment, to be discussed later, we have embarked on a more thorough investigation of the electric and magnetic properties at macroscopic wavelengths $1/q$ ($q \ll$ Fermi wave number k_F) and low frequencies ω ($\hbar\omega \ll$ Fermi energy ϵ_F) of simple metals with a spherical Fermi surface (the case of non-spherical surfaces, with *isotropic* interaction, is discussed in Refs. 11 and 12).

In this paper we consider only the magnetic properties of these paramagnetic materials where the effects of incipient ferromagnetism are negligible. That is, we shall examine those collective excitations (spin waves) involving the magnetization density within the framework of the Landau theory—which imposes the limits on (ω, q) described above.

In the work of Landau, Silin, and all subsequent authors, the linear response in this regime of an infinite Fermi liquid was studied by solving the Fourier-transformed Boltzmann-Landau equation for the distribution function $\delta f(\mathbf{q}, \omega)$. Here we adopt a different approach. We shall solve in real space time for $\delta f(\mathbf{r}, t)$ by use of Chamber's[15] integral formulation of the transport equation by extending his results to the case of the interacting electron gas. This method, emphasizing in a clear manner the physical processes described, has found many applications in solid-state physics.[16–18] In using it we shall lay stress on the motion of the individual particles, a salient feature in Landau's quasiparticle theory. Not only for this reason have we adopted Chamber's method, but also because our results are in a form that might be applied directly to transient phenomena in a bounded system.

We shall also consider the effects of collisions in a more general fashion than has been attempted before. In particular, we do not care to limit ourselves to the case of large-angle scattering. However, we shall restrict ourselves to the relaxation-time model.

In summary, we present in this paper an algorithm for discussing all of the collective magnetization modes in a simple metal when a small rf magnetic field is applied either along or normal to a static magnetic field. The results are valid for wavelengths arbitrarily long or short compared to the cyclotron radius (provided $q \ll k_F$) and for propagation at an arbitrary angle to the static field. In addition, an arbitrarily large number of the Landau scattering coefficients, as well as the dissipative scattering coefficients, are retained. Use of the algorithm enables us to catalog systematically the dispersion properties as well as the oscillator strengths of all of the waves in an infinite medium. It also permits us to study the transmission of spin waves in finite slabs. All of these results are applicable to the neutral Fermi liquid He^3 by removing the charge-dependent parameters from the results.

Conduction electron spin resonance (CESR) was first observed in reflection from a sodium surface by Griswold *et al.*[19] and treated analytically by Dyson.[20] More recently Schultz and Latham[21] were able to measure the signal transmitted by a copper slab. Later Schultz and Dunifer[5] repeated the experiment using sodium and observed, in addition to the CESR line, a complex structure of geometric resonances. The explanation for these, given very successfully by Platzman and Wolff,[22] provided the first really convincing evidence for the validity of Landau's theory. We shall reconsider their results and show how a more refined approach improving the agreement between theory and experiment permits further information on the nature of the Landau interaction to be extracted from these experiments.

In Sec. II we consider the dynamics of quasiparticles in the combined external and Landau fields—an exposition of the Landau model first suggested by Nozières.[14] In Sec. III we treat the kinetics of the particles including the effect of collisions and we also derive the integral form of the kinetic equations describing the various types of spin and nonspin processes. Subsequent sections treat the solution to these equations (Sec. V), the algorithm for determining the susceptibility (Sec. VI), and the dispersion properties and relative strengths of the collective modes (Secs. VII–IX). Then follows a comparison of the theory with the experimental results of Schultz and Dunifer.

II. QUASIPARTICLE DYNAMICS

At the basis of Landau's theory is the assumption that at low enough temperatures there exist energy states which behave like individual particles. These "quasiparticles" may be considered localized in space if we are interested in their behavior over distances long compared with atomic dimensions. Quasiparticles, characterized by momentum $\mathbf{k}$ (in units $\hbar=1$) and spin $\boldsymbol{\sigma}$, constitute the Fermi liquid which is described at point $\mathbf{r}$ and time t by the distribution $f(\boldsymbol{\sigma},\mathbf{k},\mathbf{r},t)$. The latter, an element of a 2×2 matrix in spin space, is the Wigner[23] distribution function. The related quasiparticle energy is the element $\mathcal{H}(\boldsymbol{\sigma})$ of an energy matrix. We shall use $f(\boldsymbol{\sigma})$ to discuss the dynamics and will consider its intrinsic behavior in the section on kinetics.

It is convenient to introduce certain spin averages. We have the number density

$$n=\tfrac{1}{2}\,\mathrm{Tr}f=\tfrac{1}{2}\sum_{s,s'}f_{ss'}\delta_{s's}=\tfrac{1}{2}(f_{\uparrow\uparrow}+f_{\downarrow\downarrow}) \tag{2.1}$$

and the corresponding spin density

$$S_i\equiv S(\sigma_i)=\tfrac{1}{2}\,\mathrm{Tr}(\sigma_i f)=\tfrac{1}{2}\sum_{s,s'}(\sigma_i)_{ss'}f_{s's}. \tag{2.2}$$

The components σ_i of the spin vector $\boldsymbol{\sigma}$ are, initially, the Pauli matrices which describe rotations with respect to the static magnetic field $H\hat{z}$ (the circumflex indicates a unit vector). We have, initially, $\boldsymbol{\sigma}=(\sigma_+,\sigma_-,\sigma_z)$, where $\sigma_\pm=(1/\sqrt{2})(\sigma_x\mp i\sigma_y)$. That is,

$$\sigma_-=\begin{pmatrix}0 & \sqrt{2}\\ 0 & 0\end{pmatrix},\quad \sigma_+=\begin{pmatrix}0 & 0\\ \sqrt{2} & 0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Consequently,

$$S_+=(1/\sqrt{2})f_{\downarrow\uparrow},\qquad S_-=(1/\sqrt{2})f_{\uparrow\downarrow},\qquad S_z=\tfrac{1}{2}(f_{\uparrow\uparrow}-f_{\downarrow\downarrow}). \tag{2.3}$$

Similarly we may define the spin-independent energy

$$\epsilon=\tfrac{1}{2}(\mathcal{H}_{\uparrow\uparrow}+\mathcal{H}_{\downarrow\downarrow}) \tag{2.4}$$

and the spin-dependent counterparts

$$\mathcal{H}_+=(1/\sqrt{2})\mathcal{H}_{\downarrow\uparrow},\quad \mathcal{H}_-=(1/\sqrt{2})\mathcal{H}_{\uparrow\downarrow},\quad \mathcal{H}_z=\tfrac{1}{2}(\mathcal{H}_{\uparrow\uparrow}-\mathcal{H}_{\downarrow\downarrow}). \tag{2.5}$$

The matrix character of the spin vector $\boldsymbol{\sigma}$ will hereafter be assumed. Apart from a portion of Sec. III we shall not write the spin subscripts explicitly. Thus $\boldsymbol{\sigma}$ (or $\boldsymbol{\sigma}_{rs}$) denotes element (rs) in the matrix of the component σ_i parallel to $\boldsymbol{\sigma}$. Note also that for brevity we write the matrix $f(\boldsymbol{\sigma}_{rs})\equiv f_{rs}(\boldsymbol{\sigma})$ and the scalar $f(\sigma_i)\equiv f_i$.

When a small rf magnetic field δH ($|\delta H|\ll H$) is applied,[24] the magnetic moments precessing uniformly in the static field H gain energy. This perturbation sets up spatial and temporal fluctuations of both the spin density and also (via the associated electric field) the number density. The total field experienced by a quasiparticle changes as a result. Landau expressed the relationship between the increment in energy $\delta\mathcal{H}(\boldsymbol{\sigma})$ and the change in the distribution $\delta f(\boldsymbol{\sigma},\mathbf{k},\mathbf{r},t)$ in the form of the integral equation

$$\delta\mathcal{H}(\boldsymbol{\sigma},\mathbf{k},\mathbf{r},t)=\tfrac{1}{2}\,\underset{\sigma'}{\mathrm{Tr}}\sum_{\mathbf{k}'}\mathcal{F}(\mathbf{k},\boldsymbol{\sigma};\mathbf{k}',\boldsymbol{\sigma}')\times\delta f(\boldsymbol{\sigma}',\mathbf{k}')+\tfrac{1}{2}\gamma\boldsymbol{\sigma}\cdot\delta\mathbf{H}, \tag{2.6}$$

where $\mathcal{F}$ is the Landau scattering matrix. As we shall discuss later, in an isotropic system from which spin-orbit coupling is absent, the spin dependence is given by

$$\mathcal{F}(\mathbf{k},\boldsymbol{\sigma};\mathbf{k}',\boldsymbol{\sigma}')=\mathcal{A}(\mathbf{k},\mathbf{k}')+\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\mathcal{B}(\mathbf{k},\mathbf{k}'), \tag{2.7}$$

where $\mathcal{A}$ and $\mathcal{B}$ are experimentally determinable functions. The sum in (2.6) is to be carried out over the continuum of momenta up to the Fermi momentum k_F. The last term on the right of (2.6) is just the free-electron energy arising from its magnetic moment $\gamma\boldsymbol{\sigma}=g\mu_B\boldsymbol{\sigma}$ in the driving field δH, where μ_B is the Bohr magneton and $g\approx 2$.

We shall classify the different types of scattering in the following way.

(i) Spin-independent density fluctuations $\delta n(\mathbf{k})$ cause the quasiparticle energy to change, as a result of scattering, by an amount

$$\delta U(\mathbf{k})=\tfrac{1}{2}\sum_{\mathbf{k}'}\mathcal{A}(\mathbf{k},\mathbf{k}')\delta n(\mathbf{k}'), \tag{2.8}$$

and we may regard δU as the spin-independent Landau field.

(ii) Spin-density fluctuations $\delta\mathbf{S}(\mathbf{k})$, depending on the scattering of the spins with the external field, bring about an energy increment

$$\tfrac{1}{2}\gamma\boldsymbol{\sigma}\cdot(\delta\mathbf{H}+\delta\mathbf{H}^L)\equiv\tfrac{1}{2}\gamma\boldsymbol{\sigma}\cdot\delta\mathbf{H}^*(\mathbf{k},\mathbf{r},t), \quad (2.9)$$

where the Landau field $\delta\mathbf{H}^L(\boldsymbol{\sigma})$ is given by

$$\tfrac{1}{2}\gamma\delta\mathbf{H}^L(\boldsymbol{\sigma},\mathbf{k},\mathbf{r},t)=\sum_{\mathbf{k}'}\mathcal{B}(\mathbf{k},\mathbf{k}')\delta\mathbf{S}(\boldsymbol{\sigma},\mathbf{k}',\mathbf{r},t). \quad (2.10)$$

In other words, the effective field $\delta\mathbf{H}^*$ is the external field $\delta\mathbf{H}$ plus the Landau field $\delta\mathbf{H}^L$ due to interactions. Phrased in this manner, Nozières[14] pointed out the similarity between the Landau theory and the molecular field theory of Weiss. The Landau field varies from point to point but, unlike the Weiss field, it also depends on the momentum of the particle experiencing that field. This momentum-dependent force is the source of the wealth of phenomena predictable from the theory. We shall be concerned with type (ii) of the two types of effects described above. However, in the interests of uniformity we shall develop simultaneously the kinetics and dynamics of both processes.

The effects of interaction do not arise solely as a result of the oscillatory electromagnetic field causing (self-consistently) density fluctuations. An enhancement of the free-fermion Pauli magnetization is the feature associated with static magnetic field applied. In such situations the spin vector $\boldsymbol{\sigma}$ is aligned parallel or antiparallel to the field $H\hat{z}$, corresponding to the spin-up and -down electrons, respectively. The difference in population is just

$$\Delta S_z=\tfrac{1}{2}\,\mathrm{Tr}(\sigma_z f^0(\boldsymbol{\sigma}))=\tfrac{1}{2}[f^0(\mathcal{H}_{\uparrow\uparrow}{}^0)-f^0(\mathcal{H}_{\downarrow\downarrow}{}^0)],$$

where f^0 is the Fermi-Dirac distribution and the static quasiparticle energy is

$$\mathcal{H}_{\uparrow\uparrow}{}^0=\epsilon+\Delta\mathcal{H}, \qquad \mathcal{H}_{\downarrow\downarrow}{}^0=\epsilon-\Delta\mathcal{H},$$

where ϵ is the kinetic energy. For fields $H\ll\epsilon_F/\gamma$ or $\kappa T/\gamma$,

$$\begin{aligned}\Delta S_z&=\tfrac{1}{2}[(f^0(\epsilon)+\Delta\mathcal{H}(\partial f^0/\partial\epsilon)+\cdots)\\&\qquad-(f^0(\epsilon)-\Delta\mathcal{H}(\partial f^0/\partial\epsilon)+\cdots)]\\&=\Delta\mathcal{H}(\partial f^0/\partial\epsilon)+O(\Delta^2),\end{aligned} \quad (2.11)$$

where $\Delta\sim\gamma H/\epsilon_F$ at very low temperatures $\kappa T\ll\epsilon_F$. Just as in (2.6), we have

$$\Delta\mathcal{H}=\sum_{\mathbf{k}'}\mathcal{B}(\mathbf{k},\mathbf{k}')\Delta S_z(\mathbf{k}')+\tfrac{1}{2}\gamma H\equiv\tfrac{1}{2}\gamma H^*, \quad (2.12)$$

thus defining an effective static field H^*. The latter is therefore related to the applied field H by the equation

$$H^*\left(1-\sum_{\mathbf{k}'}\mathcal{B}(\mathbf{k},\mathbf{k}')\frac{\partial f^0}{\partial\epsilon(\mathbf{k}')}\right)=H. \quad (2.13)$$

Up to now we have described the changes in energy due to the combined effects of the magnetic and Landau fields. However, we have not yet described the static equilibrium energy with respect to which the time-dependent changes take place. Functionally, it is defined as the energy

$$\mathcal{H}=\epsilon(\mathbf{k}-(e/c)\mathbf{A}_0) \quad (2.14)$$

of a particle-hole pair in the static field

$$\mathbf{H}=\nabla\times\mathbf{A}_0(\mathbf{r}), \quad (2.15)$$

where $\mathbf{A}_0$ is a vector potential and $-e$ is the electronic charge. *All changes in quasiparticle energy result from time-dependent fluctuations about this equilibrium energy.*

With the application of a time-dependent potential $\delta\mathbf{A}(\mathbf{r},t)$ the above energy changes (if no spin-flip occurs) by an amount

$$\partial\epsilon=\mathcal{H}(\mathbf{p})-\mathcal{H}(\mathbf{k})=-(e/c)\delta\mathbf{A}\cdot(\partial\epsilon/\partial\mathbf{p})+O(\delta^2) \quad (2.16)$$

because the total momentum conjugate to position $\mathbf{r}$ is then

$$\mathbf{p}=\mathbf{k}-(e/c)(\mathbf{A}_0+\delta\mathbf{A}). \quad (2.17)$$

By choice of gauge for the electromagnetic potential the rf electric field is

$$\delta\mathbf{E}(\mathbf{r},t)=-c^{-1}[\partial\delta\mathbf{A}(\mathbf{r},t)/\partial t], \quad (2.18)$$

and the rf magnetic field is

$$\delta\mathbf{H}(\mathbf{r},t)=\nabla\times\delta\mathbf{A}(\mathbf{r},t). \quad (2.19)$$

We have already discussed the spin-dependent effects induced by the latter. We see that so far as energy changes are concerned, the electric field enters only through the spin scalar $\delta\epsilon$ and hence will modify the singlet energy in the Landau field (2.8),

$$\delta U^*=\delta U+\delta\epsilon. \quad (2.20)$$

Summarizing, the local energy of a quasiparticle is equal to its static equilibrium energy ϵ plus the energy gained from interaction with the rf electromagnetic and Landau fields. We have a triplet of states with energies

$$\mathcal{H}_i=\epsilon+\tfrac{1}{2}\gamma\boldsymbol{\sigma}_i\cdot\delta\mathbf{H}^* \quad (2.21)$$

resulting from spin and momentum scattering and a singlet

$$\mathcal{H}_0=\epsilon+\delta U^* \quad (2.22)$$

arising from momentum scattering only.

From these Hamiltonians we derive directly the equations of motion. Correct to zeroth order in δ we have

$$\begin{aligned}\dot{\mathbf{r}}&=\frac{\partial\epsilon(\mathbf{k}-(e/c)\mathbf{A}_0)}{\partial\mathbf{p}}\\&=(\partial\epsilon/\partial\mathbf{k})+O(\delta)\equiv\mathbf{k}/m^*, \qquad (2.23)\\\dot{\mathbf{p}}&=-(\partial\epsilon/\partial\mathbf{r})+O(\delta),\end{aligned}$$

or, from (2.15), (2.17), and (2.23),

$$\begin{aligned}\dot{\mathbf{k}}&=\dot{\mathbf{p}}-(e/c)(d/dt)(\mathbf{A}_0)+O(\delta)\\&=-(e/c)(\partial\mathbf{A}_0/\partial\mathbf{r})\cdot\dot{\mathbf{r}}+O(\delta)\\&=-(e/m^*c)(\mathbf{k}\times\mathbf{H})+O(\delta)\equiv-\omega_c(\mathbf{k}\times\hat{z}),\end{aligned} \quad (2.24)$$

showing that a quasiparticle is accelerated by a Lorentz force in a direction normal to its momentum and to the magnetic field.

The last equality in (2.24) is a definition of the effective mass m^*. In this definition are incorporated the many-body effects of the electron-phonon interaction. The mass m^* is related to the crystalline mass by the Landau interaction and so should form part of the present treatment. However, the crystalline mass is not known accurately from either experiments or from band calculations. Therefore we shall simply consider m^* as a phenomenological parameter. It should be remembered that the definition (2.24) is valid only for spherical Fermi surfaces and in such cases it determines the density of states $\mathfrak{N}_F$ (for a single spin) at $\epsilon=\epsilon_F$

$$\mathfrak{N}_F=m^*k_F/2\pi^2. \tag{2.25}$$

It is also equal to the cyclotron mass, as measured by Azbel-Kaner resonance,[25] but only in the case of ellipsoidal Fermi surfaces—another example of the high degeneracy of the isotropic system.

One important dynamical effect, especially relevant to the spin waves, has been left unmentioned. The spin vector $\boldsymbol{\sigma}(t)$ behaves like a source of angular momentum precessing in the total magnetic field[14]:

$$\partial\boldsymbol{\sigma}/\partial t\,|_{\text{precession}}=-\gamma\boldsymbol{\sigma}\times(\mathbf{H}^*+\delta\mathbf{H}^*)$$
$$=-\gamma\boldsymbol{\sigma}\times\mathbf{H}^*+O(\delta). \tag{2.26}$$

This equation could have been derived from the Hamiltonian (2.21) by regarding $\boldsymbol{\sigma}$ as the spin component of the generalized momenta $(\mathbf{p}_\sigma, \mathbf{p}_k)$ the coordinates to which they are conjugated being $(\boldsymbol{\psi}, \mathbf{r})$, where the $\boldsymbol{\psi}$ is the set of angles generated from the Pauli rotation matrices.

It is clear, from (2.26), that

$$\sigma_z=\text{const} \tag{2.27}$$

and also

$$\sigma_\pm(t)=\sigma_\pm(t_0)\exp[\mp i\omega_L^*(t-t_0)], \tag{2.28}$$

where

$$\omega_L^*=\gamma H^*=\tfrac{1}{2}g(eH^*/mc) \tag{2.29}$$

is the Larmor spin-precession frequency in the field H^*. Returning with this result to (2.2) we see that the spin-density precesses

$$\delta S_\pm(t)\,|_{\text{precession}}=\delta S_\pm(t_0)\exp[\mp i\omega_L^*(t-t_0)]. \tag{2.30}$$

We could return to consider first-order effects in the equations of motion and would find that the particles were accelerated by the electric field, Lorentz forces, and spatial gradients of the Landau field. However, for the purposes of calculating transport coefficients we shall require explicit forms only for the unperturbed particle trajectories $[\mathbf{k}=\mathbf{k}(t),\ \mathbf{r}=\mathbf{r}(t)$, and $\boldsymbol{\sigma}=\boldsymbol{\sigma}(t)]$. As we shall see, the function of the first-order terms in the Hamiltonian is to drive the collective waves through the explicit time dependence of the Hamiltonian.

III. QUASIPARTICLE KINETICS

We examined the dynamics of a single quasiparticle by looking at its Hamiltonian $\mathcal{H}(f(\boldsymbol{\sigma},\mathbf{k},\mathbf{r},t))$. In order to obtain results explicit in $(\mathbf{r},t)$ we need to examine the kinetics of the quasiparticle population change $\delta f(\boldsymbol{\sigma})$. One procedure for doing this is to solve a kinetic equation: the Boltzmann equation as adapted by Landau and Silin. We shall derive our results directly from an integral formulation describing the scattering process.

Let $f(\boldsymbol{\sigma})$ be the perturbed distribution in the presence of fields and let $f^0(\mathcal{H}(\boldsymbol{\sigma})-\delta\mathcal{H}(\boldsymbol{\sigma}))$ be its value at equilibrium where f^0 is the Fermi-Dirac distribution. Thus the change in population is given by [see (2.21) and (2.22)]

$$f(\boldsymbol{\sigma})=f^0(\mathcal{H}(\boldsymbol{\sigma})-\delta\mathcal{H}(\boldsymbol{\sigma}))+\delta f(\boldsymbol{\sigma})$$
$$=f^0(\mathcal{H}(\boldsymbol{\sigma}))+\delta\bar{f}(\boldsymbol{\sigma})+O(\delta^2), \tag{3.1}$$

where the quantity

$$\delta\bar{f}(\boldsymbol{\sigma})=\delta f(\boldsymbol{\sigma})-(\partial f^0/\partial\epsilon)\delta\mathcal{H}(\boldsymbol{\sigma}) \tag{3.2}$$

is the deviation of $f(\boldsymbol{\sigma})$ from its local equilibrium value $f^0(\mathcal{H}(\boldsymbol{\sigma}))$. Differentiating (3.1) with respect to t and taking the various spin averages (2.1)–(2.5), we obtain correct to order δ

$$\frac{dS_i}{dt}=\frac{df^0}{d\epsilon}\boldsymbol{\sigma}_i\cdot\frac{\partial}{\partial t}(\tfrac{1}{2}\gamma\delta H^*)\Big|_{\mathbf{k},\mathbf{r}}+\frac{d}{dt}\delta\bar{S}_i, \tag{3.3}$$

$$\frac{dn}{dt}=\frac{df^0}{d\epsilon}\frac{\partial}{\partial t}(\delta U^*)\Big|_{k,r}+\frac{d\delta\bar{n}}{dt}, \tag{3.4}$$

where

$$\delta\bar{S}_i=\tfrac{1}{2}\operatorname{Tr}(\sigma_i\delta\bar{f}), \tag{3.5}$$

$$\delta\bar{n}=\tfrac{1}{2}\operatorname{Tr}(\delta\bar{f}). \tag{3.6}$$

These equations describe the rates of population change of the various components of spin as a result of particles accelerating in the Landau and applied fields. Counteracting such processes are collisional effects. As described by Pines and Nozières[26] the rate of change of the distribution $f(\mathbf{k})$, in the absence of spin, is

$$df/dt=\sum_{\mathbf{k}'}\mathcal{W}(\mathbf{k},\mathbf{k}')\delta(\mathcal{H}(\mathbf{k}')-\mathcal{H}(\mathbf{k}))$$
$$\times\{f(\mathcal{H}(\mathbf{k}'))[1-f(\mathcal{H}(\mathbf{k}))]-f(\mathcal{H}(\mathbf{k}))[1-f(\mathcal{H}(\mathbf{k}'))]\},$$

where $\mathcal{W}$ is the scattering matrix for real processes between the states $\mathbf{k}$ and $\mathbf{k}'$. Notice that the local energy $\mathcal{H}(\mathbf{k},\mathbf{r})$—rather than the equilibrium energy $\epsilon(\mathbf{k})=\mathcal{H}(\mathbf{k})-\delta\mathcal{H}(\mathbf{k})$—is conserved by the collisions. We recognize the first term in the curly brackets as the "scatter-out" contribution. This form of the collision integral accounts correctly for scattering by impurities, the most important process in metals at low temperatures.[26]

The generalization to include spin is straightforward. With the distribution function f a function of $\boldsymbol{\sigma}$ and $\mathbf{k}$, we have

$$df(\boldsymbol{\sigma},\mathbf{k})/dt\,|_{\text{collision}} = \sum_{\mathbf{k}',\boldsymbol{\sigma}'} \mathcal{W}(\mathbf{k},\boldsymbol{\sigma};\mathbf{k}',\boldsymbol{\sigma}')\,\delta(\mathcal{H}(\boldsymbol{\sigma}',\mathbf{k}')-\mathcal{H}(\boldsymbol{\sigma},\mathbf{k}))\times\{f(\boldsymbol{\sigma}',\mathbf{k}')[1-f(\boldsymbol{\sigma},\mathbf{k})]-f(\boldsymbol{\sigma},\mathbf{k})[1-f(\boldsymbol{\sigma}',\mathbf{k}')]\}. \quad (3.7)$$

From (3.1), we have

$$f(\boldsymbol{\sigma}_{rs})=f^0(\mathcal{H}(\boldsymbol{\sigma}_{rs}))\delta_{rs}+\delta\bar{f}(\boldsymbol{\sigma}_{rs})$$

and

$$\mathcal{H}(\boldsymbol{\sigma}_{rs},\mathbf{k})=\epsilon(\mathbf{k})+\delta\mathcal{H}(\boldsymbol{\sigma}_{rs},\mathbf{k}).$$

We then obtain the linearized equation

$$df(\boldsymbol{\sigma}_{rs},\mathbf{k})/dt\,|_{\text{collision}}=\sum_{\mathbf{k}'r's'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\times\mathcal{W}(\mathbf{k},\boldsymbol{\sigma}_{rs};\mathbf{k}',\boldsymbol{\sigma}_{s'r'})\,\delta\bar{f}(\boldsymbol{\sigma}_{r's'},\mathbf{k}')-\tau^{-1}\delta\bar{f}(\boldsymbol{\sigma}_{rs},\mathbf{k}), \quad (3.8)$$

where we have the scatter-out rate defined as

$$1/\tau=\sum_{\mathbf{k}'}\operatorname*{Tr}_{\sigma'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\mathcal{W}(\mathbf{k},\boldsymbol{\sigma};\mathbf{k}',\boldsymbol{\sigma}') = \sum_{\mathbf{k}',r's'}\delta_{r's'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\mathcal{W}(\mathbf{k},\boldsymbol{\sigma}_{rr};\mathbf{k}',\boldsymbol{\sigma}_{s'r'}). \quad (3.9)$$

The result (3.9), written explicitly with spin subscripts for clarity, follows from our assumption about the rotational invariance of the system. That is,

$$\sum_{\mathbf{k}',r'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\mathcal{W}(\mathbf{k},\boldsymbol{\sigma}_{rs};\mathbf{k}',\boldsymbol{\sigma}_{r'r'}) = \mathcal{W}_1(\mathbf{k})\delta_{rs}+\mathbf{k}\cdot\boldsymbol{\sigma}_{rs}\mathcal{W}_2(\mathbf{k}),$$

since $\mathcal{W}$ is a scalar. Because of parity inversion it is clear that $\mathcal{W}_2$ must be zero. Further, we shall be interested in cases where $\epsilon(\mathbf{k})=\epsilon_F$, so that we may regard τ as a constant on this constant energy surface.

There are two types of collision processes to be considered in an isotropic system. One of these results in the change of orbital momentum $\mathbf{k}$ without any change in the spin $\boldsymbol{\sigma}$ of the quasiparticle. The other concerns the flipping of a spin resulting principally from spin-orbit coupling.[27] Neglecting the anisotropy of the latter we may then write

$$\mathcal{W}(\mathbf{k},\boldsymbol{\sigma}_{rs};\mathbf{k}',\boldsymbol{\sigma}_{s'r'})=W(\mathbf{k},\mathbf{k}')\delta_{rr'}\delta_{ss'} +[1/2\mathfrak{N}(\epsilon)\tau_s]\delta_{rs}\delta_{r's'}, \quad (3.10)$$

where τ_s is the spin-lattice relaxation time and $\mathfrak{N}(\epsilon)$ is the density of states [cf. (2.25)]. This expression can also be written

$$\mathcal{W}(\mathbf{k},\boldsymbol{\sigma}_{rs};\mathbf{k}',\boldsymbol{\sigma}_{s'r'})=\tfrac{1}{2}\{(W+1/\mathfrak{N}\tau_s)\delta_{rs}\delta_{s'r'}+W\boldsymbol{\sigma}_{rs}\cdot\boldsymbol{\sigma}_{s'r'}\},$$

which is similar to the form adopted for $\mathfrak{F}$ [see (2.7)]. We can thus interpret $\mathfrak{N}W$ as the scattering rate for the particle-hole system in the spin-triplet state and $(\mathfrak{N}W+1/\tau_s)$ as the rate in the singlet state.

The collision terms differ from those of Platzman and Wolff[22] in several respects. Firstly, they have neglected the dependence of the scattering on the deflection angle $\cos^{-1}(\hat{k}\cdot\hat{k}')$. Secondly, the spin-lattice relaxation time τ_s, introduced here in the same general scheme as the orbital relaxation time, plays a role in the spin-wave dispersion relations slightly different from that in the Platzman-Wolff work. Lastly, the collision terms there are proportional to the deviation δf from *true* equilibrium, whereas they should be proportional to the deviation $\delta\bar{f}$ from *local* equilibrium.[26] While this is an error its associated magnitude (Sec. X) is not very great in Na or K. As regards the two other discrepancies, the inclusion of angle dependence has so far not been found necessary for comparison with experimental data[6] and, since $\tau_s \gg \tau$, little error was made by Platzman and Wolff in this respect.

The treatment accorded here to the collision terms is strictly valid only for elastic impurity scattering. However, for temperatures much less than the Debye temperature the scattering from phonons is correctly taken into account if the τ_l are made temperature dependent.

If the only loss mechanisms are the relaxation processes just described then we may write

$$df(\boldsymbol{\sigma})/dt=df(\boldsymbol{\sigma})/dt\,|_{\text{collision}}, \quad (3.11)$$

which is our fundamental kinetic equation. Hence, by using (3.1)–(3.4) and (3.8)–(3.10) the linearized kinetic equations for the various spin channels follow:

$$\frac{d}{dt}\delta\bar{n}(\mathbf{k})+\frac{\partial\delta U^*}{\partial t}\frac{df^0}{d\epsilon}=\sum_{\mathbf{k}'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\times[W(\mathbf{k},\mathbf{k}')+(\mathfrak{N}\tau_s)^{-1}]\delta\bar{n}(\mathbf{k}')-\frac{\delta\bar{n}(\mathbf{k})}{\tau}, \quad (3.12)$$

$$\frac{d}{dt}\delta\bar{S}_z(\mathbf{k})+\tfrac{1}{2}\gamma\sigma_z\frac{\partial\delta H_z^*}{\partial t}\frac{df^0}{d\epsilon}=\sum_{\mathbf{k}'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\times[W(\mathbf{k},\mathbf{k}')+(\mathfrak{N}\tau_s)^{-1}]\delta\bar{S}_z(\mathbf{k}')-\frac{\delta\bar{S}_z(\mathbf{k})}{\tau}, \quad (3.13)$$

$$\frac{d}{dt}\delta\bar{S}_\pm(\mathbf{k})+\tfrac{1}{2}\gamma\sigma_\mp\frac{\partial}{\partial t}\delta H_\pm^*\frac{df^0}{d\epsilon}=\sum_{\mathbf{k}'}W(\mathbf{k},\mathbf{k}')\times\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\delta\bar{S}_\pm(\mathbf{k}')-\frac{\delta\bar{S}_\pm(\mathbf{k})}{\tau}. \quad (3.14)$$

We note that the channels are completely decoupled as a result of the assumptions (2.7) and (3.10) about the virtual and real scattering matrices $\mathfrak{F}$ and $\mathfrak{W}$, respectively.

Let us consider Eq. (3.14). We can integrate it directly

$$\delta\bar{S}_{\pm}(\mathbf{k},\mathbf{r},t)=\delta\bar{S}_{\pm}(\mathbf{k},\mathbf{r},t_0)+\int_{t_0}^{t}dt_1\exp[-(t-t_1)/\tau]\times\Bigg[-\frac{df_0}{d\epsilon}\left(\tfrac{1}{2}\gamma\sigma_{\mp}(t_1)\frac{\partial\delta H_{\pm}^*}{\partial t_1}(\mathbf{k}(t_1),\mathbf{r}(t_1),t_1)\right)+\sum_{\mathbf{k}'(t_1)}W(\mathbf{k},\mathbf{k}')\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}(t_1)))\times\delta\bar{S}_{\pm}(\mathbf{k}'(t_1),\mathbf{r}(t_1),t_1)\Bigg],\quad(3.15)$$

where, from (2.23)

$$\mathbf{r}(T)=\int^{T}dt_2\frac{\mathbf{k}(t_2)}{m^*}-\mathbf{r}(t_0).\quad(3.16)$$

Having obtained the result above in real space and time $(\mathbf{r},t)$ rather than in Fourier space $(\mathbf{q},\omega)$, it is easy to insert initial and boundary value conditions on $\delta\bar{S}_{\pm}$. It is somewhat less easy to solve such problems since time parametrizes the momentum $\mathbf{k}$ throughout and since $\delta H_{\pm}$ is related to $\delta\bar{S}_{\pm}$ via an integral equation in $\mathbf{k}$ [cf. (2.6) and (2.7)]. Nevertheless, the result would be of great value for the study of pulse phenomena, e.g., spin echoes.[28]

If we consider the "steady-state" problem, the initial value of $\delta\bar{S}_{\pm}$ will not appear in the result. We can set $\delta\bar{S}_{\pm}(\mathbf{r}(t_0),\ t_0)=0$ and $t_0=-\infty$ in (3.15). We now suppose that the small rf magnetic field $\delta H_{\pm}$ varies as $e^{-i\omega t}$, thus imparting its timelike behavior to $\delta\bar{S}_{\pm}$. If, in addition, we suppose the Fermi liquid to be infinite or, at least, translationally invariant (on a macroscopic, not an atomic scale), then absolute positions disappear from the problem. We shall assume that $(\delta\bar{S}_{\pm},\ \delta H_{\pm}^*)\propto e^{i\mathbf{q}\cdot\mathbf{r}}$. Finally, using (2.28) and (2.30), we obtain the integral form of the kinetic equation:

$$\delta\bar{S}_{\pm}(\mathbf{k}(t),\mathbf{q},\omega)=\int_{-\infty}^{t}dt_1\times\left\{\exp i\left[\left(\omega\pm\omega_L^*+\frac{i}{\tau}\right)(t-t_1)+\mathbf{q}\cdot[\mathbf{r}(t_1)-\mathbf{r}(t)]\right]\right\}\times\left\{i\tfrac{1}{2}\omega\gamma\frac{df^0}{d\epsilon}\delta H_{\pm}^*(\mathbf{k}'(t_1))+\sum_{\mathbf{k}'}W(\mathbf{k},\mathbf{k}')\times\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\delta\bar{S}_{\pm}(\mathbf{k}'(t_1))\right\}.\quad(3.17)$$

The left-hand side still depends on time, but only implicitly through $\mathbf{k}(t)$. That explicit dependence remains can be shown by introducing $T=(t-t_1)$ in the right-hand side of (3.17), which gives

$$\delta\bar{S}_{\pm}(\mathbf{k}(t),\mathbf{q},\omega)=\int_0^{\infty}dT\times\left\{\exp i\left[\left(\omega\pm\omega_L^*+\frac{i}{\tau}\right)T+\mathbf{q}\cdot[\mathbf{r}(t-T)-\mathbf{r}(t)]\right]\right\}\times\left\{i\omega\tfrac{1}{2}\gamma\frac{df^0}{d\epsilon}\delta H_{\pm}^*(\mathbf{k}'(t-T))+\sum_{\mathbf{k}'}W(\mathbf{k},\mathbf{k}')\times\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\delta\bar{S}_{\pm}(\mathbf{k}'(t-T),\mathbf{q},\omega)\right\}.\quad(3.18)$$

For the longitudinal spin fluctuations we can prove

$$\delta\bar{S}_z(\mathbf{k}(t),\mathbf{q},\omega)=\int_0^{\infty}dT\exp i[(\omega+i/\tau)T+\mathbf{q}\cdot(\mathbf{r}(t-T)-\mathbf{r}(t))]\times\left\{i\omega\tfrac{1}{2}\gamma\frac{df^0}{d\epsilon}\delta H_z^*(\mathbf{k}'(t-T))+\sum_{\mathbf{k}'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))(W(\mathbf{k},\mathbf{k}')+(\mathfrak{N}\tau_s)^{-1})\right\},\quad(3.19)$$

while, for spin-independent changes,

$$\delta\bar{n}(\mathbf{k}(t),\mathbf{q},\omega)=\int_0^{\infty}dT\exp i[(\omega+i/\tau)T+\mathbf{q}\cdot(\mathbf{r}(t-T)-\mathbf{r}(t))]\times\left\{\frac{df^0}{d\epsilon}\frac{\partial\delta U^*}{\partial T}+\sum_{\mathbf{k}'}\delta(\epsilon(\mathbf{k})-\epsilon(\mathbf{k}'))\times(W(\mathbf{k},\mathbf{k}')+(\mathfrak{N}\tau_s)^{-1})\right\}.\quad(3.20)$$

These equations summarize the kinetic theory of a charged Landau-Fermi liquid. For analysis of transport phenomena we must examine the nature of the scattering of the quasiparticles.

IV. SCATTERING AT FERMI SURFACE

The Landau interaction matrix $\mathfrak{F}(\mathbf{k},\boldsymbol{\sigma},\mathbf{k}',\boldsymbol{\sigma}')$ describes the scattering of a quasiparticle between the primed and unprimed states. The energy change involved when interaction with the rf magnetic field takes place is of order $\gamma\delta H$. In alkali metals, if $\delta H\approx1$ Oe, $\delta=|\gamma\delta H/\epsilon_F|\approx10^{-8}$. When interaction with the static field occurs and spin exchange results, the energy ratio involved is $\Delta=|\gamma H/\epsilon_F|\approx10^{-3}$–$10^{-4}$ for $H\sim10^3$–10^4 Oe. So we see that spin scattering is localized to the surface of the equilibrium Fermi surface. Further, at low temperatures T the surface is blurred by thermal effects to a fractional depth $(\kappa T/\epsilon_F)\approx10^{-4}$ of the Fermi sea. To this degree of accuracy,

$$-df^0/d\epsilon=\delta(\epsilon-\epsilon_F).\quad(4.1)$$

Thus the integrated kinetic equations (3.18)–(3.20) really describe only the particles at the Fermi surface, where the quasiparticle concept and consequently $\mathfrak{F}$ and $\mathfrak{W}$ are well defined. For this reason these quantities should possess the group character of the Fermi surface. A spherical surface predicates that $\mathfrak{F}$ be a scalar in spin space. The most important of such scalar dependences is that of spin exchange—hence the choice (2.7). It also predicates isotropic dependence on $\mathbf{k}$ and $\mathbf{k}'$ and so on the angle between the vectors. We then have

$$\mathfrak{F}(\mathbf{k},\mathbf{k}')=\delta(\epsilon_F-\epsilon)\mathfrak{F}(\hat{k}\cdot\hat{k}'),$$

where $\mathbf{k}=k_F\hat{k}$ on the surface. So we shall expand the components $\mathcal{A}$ and $\mathcal{B}$ in the surface harmonics of a sphere:

$$\mathcal{A}(\hat{k}\cdot\hat{k}')=\frac{4\pi}{\mathfrak{N}_F}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_lY_{lm}^*(\hat{k}')Y_{lm}(\hat{k}),\quad(4.2)$$

$$\mathcal{B}(\hat{k}\cdot\hat{k}')=\frac{4\pi}{\mathfrak{N}_F}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_lY_{lm}^*(\hat{k}')Y_{lm}(\hat{k}),\quad(4.3)$$

where the density of states $\mathfrak{N}_F$ is defined in (2.25). The harmonics $Y_{lm}(\theta\phi)$ are those of Condon and Shortley[29] or Goldberger and Watson.[30] This choice of spin dependence and momentum dependence for $\mathfrak{F}$ also guarantees invariance under time inversion and space inversion. The matrix $\mathfrak{W}$ need not exhibit this dependence. However, the form chosen, (3.10), is the most general possible for collisions which preserve the independence of the scattering of the various spin channels. Otherwise the momentum dependence of the total scattering cross section exhibits spherical symmetry so that we may expand it:

$$W(\hat{k}\cdot\hat{k}')=\frac{4\pi}{\mathfrak{N}(\epsilon)}\sum_{l,m}(1/\tau_l)Y_{lm}^*(\hat{k}')Y_{lm}(\hat{k}).\quad(4.4)$$

Since quasiparticles are defined only near the Fermi surface,[1] the energy-conserving δ function in (3.18)–(3.20) should be replaced by $\delta(\epsilon_F-\epsilon(\mathbf{k}'))$ and the coefficients τ_l are now defined for scattering at the Fermi surface.

The sum over k states can be replaced by an integral over the Fermi sphere:

$$\sum_{\mathbf{k}}\frac{\delta(\epsilon_F-\epsilon(\mathbf{k}))}{\mathfrak{N}(\epsilon(\mathbf{k}))}=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\delta(\epsilon_F-\epsilon(\mathbf{k}))}{\mathfrak{N}(\epsilon(\mathbf{k}))}=\frac{1}{(2\pi)^3}\int d\epsilon\frac{m^*(\epsilon)k(\epsilon)}{\mathfrak{N}(\epsilon)}\delta(\epsilon_F-\epsilon)\int d^2\hat{k}=1.\quad(4.5)$$

From (3.9) we have

$$\tau^{-1}=\sum_{\mathbf{k}'}\delta(\epsilon_F-\epsilon(\mathbf{k}'))[W(\mathbf{k},\mathbf{k}')+(\mathfrak{N}_F\tau_s)^{-1}]=\tau_0^{-1}+\tau_s^{-1}.\quad(4.6)$$

Similarly we show from (2.13) that the effective static field is

$$H^*=H/(1+B_0).\quad(4.7)$$

Since we are concerned with excitations at the Fermi surface, it is appropriate to introduce the local density $G(\sigma_i)$:

$$\delta\bar{S}(\sigma_i,\hat{k})=(4\pi)^{1/2}\tfrac{1}{2}\gamma\delta H(\sigma_i)\delta(\epsilon_F-\epsilon)G(\sigma_i,\hat{k}),\quad(4.8)$$

where we have normalized $G(\sigma_i)$ with the driving field to a dimensionless quantity of order unity. The effective field $\delta H^*(\sigma_i,\hat{k},\mathbf{r},t)=\delta H^*(\sigma_i,\hat{k},\mathbf{q},\omega)e^{-i(\omega t-\mathbf{q}\cdot\mathbf{r})}$ experienced by a spin is given by (2.9) and (2.10). Replacing $\delta S(\sigma_i)$ by the local spin density $\delta\bar{S}(\sigma_i)$, (3.2) and (3.5), expressing this in terms of the local effective field, (2.9) and (2.10), and using (4.1), (4.5), and (4.8), we obtain the integral equations

$$\delta H^*(\sigma_i,\hat{k})=\delta H(\sigma_i)+(\mathfrak{N}_F/4\pi)\int d^2\hat{k}'\mathcal{B}(\hat{k}\cdot\hat{k}')\times[(4\pi)^{1/2}\delta H(\sigma_i)G(\sigma_i,\hat{k}')-\delta H^*(\sigma_i,k')]\quad(4.9)$$

for each spin component σ_i. Due to Landau scattering, the effective field δH^* seen by a particle varies from point to point over the Fermi surface. We expand it in surface harmonics:

$$\delta H^*(\sigma_i,\hat{k})=\sum_{l,m}\delta H_{lm}^*(\sigma_i)Y_{lm}(\hat{k}),\quad(4.10)$$

and similarly for the spin-density components

$$G(\sigma_i,\hat{k})=\sum_{l,m}G_{lm}{}^iY_{lm}(\hat{k}).\quad(4.11)$$

Substituting these expansions in (4.9) and using (4.3) we find that

$$\delta H_{lm}^*(\sigma_i)=(4\pi)^{1/2}\delta H_i\left(\frac{\delta_{l,0}}{1+B_0}+\frac{B_l}{1+B_l}G_{lm}{}^i\right)\quad(4.12)$$

after using the orthonormality condition

$$\int d^2\hat{k}\,Y_{lm}^*(\hat{k})Y_{l'm'}(\hat{k})=\delta_{ll'}\delta_{mm'}.\quad(4.13)$$

Even if the scattering is isotropic ($B_l=0$, $l>0$) we see that δH^* is enhanced by the factor $1/(1+B_0)$ over the free-particle result. Equation (3.18) now reads, after a little algebra,

$$G^{\pm}(\hat{k}(t),\mathbf{q},\omega)=\int_0^{\infty}dT\times\left\{\exp i\left[\left(\omega\pm\omega_L^*+\frac{i}{\tau}\right)T+\mathbf{q}\cdot[\mathbf{r}(t-T)-\mathbf{r}(t)]\right]\right\}\times\left\{-i\omega\sum_{l,m}Y_{lm}(\hat{k}(t-T))\left(\frac{\delta_{l,0}}{1+B_0}+\frac{B_l}{1+B_l}G_{lm}{}^{\pm}\right)+\sum_{l',m'}\tau_{l'}{}^{-1}G_{l'm'}{}^{\pm}Y_{l'm'}(\hat{k}(t-T))\right\},\quad(4.14)$$

and a similar equation holds for G^z.

V. INTEGRATION OF KINETIC EQUATIONS IN INFINITE MEDIUM

In a charged system with a static magnetic field it is natural to adopt the magnetic axis $\hat{z}$ as the direction reference. Without loss of generality we choose the direction of propagation $\hat{q}$ to lie in the x-z plane at an angle Δ to the z axis (Fig. 1). The polar and azimuthal angles θ and ϕ, respectively, give the direction of motion $\hat{k}$ of the quasiparticle. We shall scale time by the cyclotron period, that is, set $t=\phi/\omega_c$ [see (2.24)].

Since we refer only to particles on the Fermi surface $\mathbf{k}=k_F\hat{k}(\theta,\phi)$ we may write explicitly the spatial exponential in (4.14):

$$\mathbf{q}\cdot(\mathbf{r}(t-T)-\mathbf{r}(t))$$
$$=\frac{qk_F}{m^*}(\hat{x}\sin\Delta+\hat{z}\cos\Delta)\cdot\int_\phi^{\phi-\Phi}\frac{d\phi'}{\omega_c}\times(\hat{x}\sin\theta\cos\phi'+\hat{y}\sin\theta\sin\phi'+\hat{z}\cos\theta)$$
$$=\frac{qv_F}{\omega_c}\{\sin\Delta\sin\theta[\sin(\phi-\Phi)-\sin\phi]-\Phi\cos\Delta\cos\theta\},\tag{5.1}$$

where we have used (3.16) and defined the Fermi velocity

$$v_F=k_F/m^*.\tag{5.2}$$

From (4.11) we have

$$G_{lm}^{\pm}=\int d^2\hat{k}(\theta\phi)\,Y_{lm}^*(\hat{k}(\theta\phi))G^{\pm}(\hat{k}(\theta\phi)),\tag{5.3}$$

where the unit element $d^2\hat{k}=d\phi d\theta\sin\theta$. Substituting (4.14) into the right-hand side of this equation, it is clear that the resulting expression will involve terms which we shall hereafter label

$$\langle l'm'\mid K\mid lm\rangle=-\frac{i\omega}{\omega_c}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\,Y_{lm}^*(\theta\phi)\int_0^{\infty}d\Phi$$
$$\times\exp i[(\omega\pm\omega_L^*+i/\tau-qv_F\cos\Delta\cos\theta)\Phi/\omega_c]$$
$$\times\exp\{(iqv_F/\omega_c)\sin\Delta\sin\theta[\sin(\phi-\Phi)-\sin\phi]\}$$
$$\times Y_{l'm'}(\theta,\phi-\Phi).\tag{5.4}$$

In Appendix A it is shown how this "average over trajectories" is performed. We find

$$\langle lm\mid K(X,Y)\mid l'm'\rangle$$
$$=\frac{2\pi\omega}{\omega_c}\sum_{n=-\infty}^{\infty}\int_0^{\pi}d\theta\sin\theta\,C_{lm}(\theta)C_{l'm'}(\theta)\times\frac{J_n(X)J_{n+m'-m}(X)}{Y-(n+m')}\tag{5.5}$$
$$=(-1)^{m_>}\frac{2\pi^2\omega}{\omega_c}\int_0^{\pi}d\theta\sin\theta\,C_{lm}(\theta)C_{l'm'}(\theta)\times\frac{J_{Y-m_<}(X)J_{-(Y-m_>)}(X)}{\sin\pi Y},\tag{5.6}$$

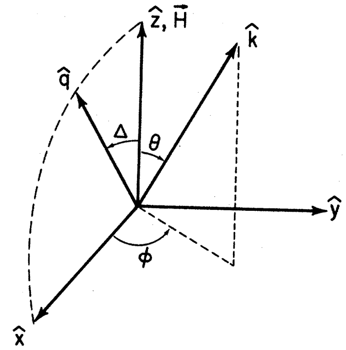

FIG. 1. System of coordinates used. **H** is the static magnetic field, $\hat{q}$ is the direction of wave propagation, and $\hat{k}$ is the direction of quasiparticle momentum. All careted quantities are unit vectors.

where $J_Y(X)$ is a Bessel function of the first kind, of order Y and argument X, and where

$$C_{lm}(\theta)=Y_{lm}(\theta,\phi)e^{-im\phi},\tag{5.7}$$
$$X(\theta)=(qv_F/\omega_c)\sin\Delta\sin\theta,\tag{5.8}$$
$$Y_{\pm}(\theta)=(\omega\pm\omega_L^*+i/\tau-qv_F\cos\Delta\cos\theta)/\omega_c,\tag{5.9}$$

$m_>=$ greater of (m,m'), $m_<=$ lesser of (m,m').

We also note that

$$\langle l'm'\mid K\mid lm\rangle=\langle lm\mid K\mid l'm'\rangle.\tag{5.10}$$

The solution to the integral equation (4.14) is given in (4.11) in terms of the coefficients $G_{lm}^{\pm}$, where the latter satisfy the infinite set of coupled algebraic equations

$$G_{lm}^{\pm}=\sum_{l',m'}[B_{l'}/(1+B_{l'})]\langle l'm'\mid K(X,Y_{\pm})\mid lm\rangle G_{l'm'}^{\pm}$$
$$+\langle 00\mid K\mid lm\rangle/(1+B_0)+\sum_{l'',m''}(i/\omega\tau_{l''})\times\langle l''m''\mid K\mid lm\rangle G_{l''m''}^{\pm}.\tag{5.11}$$

In an entirely analogous fashion we solve the kinetic equation (3.19) for $\delta\bar{S}_z$ and find that the corresponding G_{lm}^z are given by

$$G_{lm}^z=\sum_{l'm'}[B_{l'}/(1+B_{l'})]\langle l'm'\mid K(X,Y_z)\mid lm\rangle G_{l'm'}^z$$
$$+\frac{\langle 00\mid K\mid lm\rangle}{1+B_0}+i\sum_{l'',m''}\left(\frac{1}{\omega\tau_{l''}}+\frac{\delta_{l'',0}}{\omega\tau_s}\right)\times\langle l''m''\mid K\mid lm\rangle G_{l''m''}^z,\tag{5.12}$$

where all quantities are as defined earlier except that here

$$Y(\theta)=Y_z(\theta)=(\omega+i/\tau-qv_F\cos\Delta\cos\theta)/\omega_c,\tag{5.13}$$

which is independent of the enhanced spin-precession frequency ω_L^*. We shall not consider the remaining kinetic equation (3.20) in this paper.

VI. SUSCEPTIBILITY TENSOR; GENERAL ALGORITHM

In this, as in all transport theories, the physical entity investigated requires calculating not the distribution function itself but rather integrated moments of the distribution. Here we are interested in spin-density phenomena, which involve the zeroth moment with respect to momentum. They are characterized by the susceptibility tensor χ which, in the case of the rf magnetic field $\delta\mathbf{H}$ polarized either parallel or perpendicular to the static field $\mathbf{H}$, is diagonal. Defining χ in terms of the magnetic induction $\delta\mathbf{H}$, the transverse components are

$$\chi_{\pm}(\mathbf{q},\omega)=M_{\pm}(\mathbf{q},\omega)/\delta H_{\pm}, \tag{6.1}$$

and the longitudinal component is

$$\chi_{z}(\mathbf{q},\omega)=M_{z}(\mathbf{q},\omega)/\delta H_{z}. \tag{6.2}$$

The magnetization $M(\sigma_i,\mathbf{q},\omega)$ is given by

$$\begin{aligned}M(\sigma_i,\mathbf{q},\omega)&=\gamma\sum_{\hat{k}'}\delta S(\sigma_i,\hat{k}',\mathbf{q},\omega)\\&=\frac{\gamma\mathfrak{N}_F}{4\pi}\int d^2\hat{k}'(4\pi)^{1/2}\tfrac{1}{2}\gamma\delta H(\sigma_i)\\&\quad\times\left(G(\sigma_i,\hat{k}',\mathbf{q},\omega)-\frac{\delta H^*(\sigma_i,\hat{k}',\mathbf{q},\omega)}{(4\pi)^{1/2}\delta H(\sigma_i)}\right)\\&=\frac{\gamma^2}{1+B_0}\delta H(\sigma_i)2\mathfrak{N}_F[G_{00}(\sigma_i,\mathbf{q},\omega)-1],\end{aligned}$$

where we have used (4.11) and (4.12) for each component σ_i. From (6.1) and (6.2), the susceptibility components are

$$\chi_{+-z}(\mathbf{q},\omega)=\chi_0^*(G_{00}^{+-z}(\mathbf{q},\omega)-1), \tag{6.3}$$

with

$$\chi_0^*=2\mathfrak{N}_F[\gamma^2/(1+B_0)]\equiv\chi_0/(1+B_0). \tag{6.4}$$

The factor of 2 arises from the definition of $\mathfrak{N}_F$, the density of states for a single spin. We see that the static susceptibility χ_0^* is enhanced by the factor $1/(1+B_0)$ (>1 for alkali metals) over the free-electron value χ_0. Also, the dynamic part of χ is altered as a result of the Landau interaction: The free-electron result is recovered by setting $B_l=0$ ($l\geq0$) in (5.11) or (5.12).

The use of (6.3) (which defines the relevant component of the susceptibility) together with (5.11) or (5.12) which give $G_{00}^{\pm}$ or G_{00}^{z}, where the latter quantities are defined through the matrix elements given in (5.6)–(5.9) and (5.13) forms a general algorithm for the study of spin waves in paramagnetic Fermi liquids. In deriving these results at no stage (i) did we limit ourselves to the number of Landau parameters B_l or collision coefficients τ_l included, nor (ii) did we restrict the wavelength as characterized by the parameter qv_F/ω_c, nor (iii) choose a specific direction of propagation.

Retaining an infinite number of the B_l and τ_l means that an infinite set of coupled equations (5.11) or (5.12) has to be solved. However, if Landau's theory is meaningful when applied to the alkali metals, for instance, then we are at liberty to assume that the B_l decrease rapidly with increasing l so that we may truncate the system of equations. We shall deal likewise with the τ_l.

For arbitrary direction of propagation and including an arbitrary number of the B_l and τ_l, we may discuss without recourse to numerical methods the spin waves at wavelengths long ($qR_c\ll1$) and short ($qR_c\gg1$) compared with the cyclotron radius

$$R_c=v_F/\omega_c. \tag{6.5}$$

At intermediate wavelengths $qR_c\sim1$ we must perform numerical calculations. Examples of such results will indicate the practicality of the algorithm.

VII. LONG-WAVELENGTH REGIME $qR_c\ll1$

The transverse components of the susceptibility will occupy the greater part of our attention since the results for the longitudinal case, as we saw earlier, may then be obtained by a simple specialization to that case. Furthermore, we shall consider explicitly only the element $\chi_-(\mathbf{q},\omega)$; the element $\chi_+(\mathbf{q},\omega)$ is identical if we replace $-\omega_L^*$ by $+\omega_L^*$.

Let us take the system of equations (5.11), dropping all spin-direction superscripts, and truncate it at some $l=L$, i.e., set B_l, $\tau_l=0$, $l\geq L$. Retaining the same (arbitrarily large) number of the B_l as the τ_l, it is convenient to introduce the notation

$$\beta_l=B_l/(1+B_l)+(1/\omega\tau_l). \tag{7.1}$$

When the Fermi liquid is completely uniform and $q=0$ we may, by use of the properties of Bessel functions (Appendix B), show [from (5.5) or (5.6)] that

$$\langle l'm'\mid K\mid lm\rangle=\frac{\omega}{\omega+i/\tau-\omega_L^*-m\omega_c}\delta_{ll'}\delta_{mm'}. \tag{7.2}$$

In this limit the matrix of equations (5.11) becomes diagonal. By means of Cramer's rule we can express G_{00} as the ratio N/D of two determinants N, D. Then, in (6.3), χ_- has singularities at $D=0$. Inspection of this determinant shows that in the uniform medium the normal modes are given by

$$\beta_l\langle lm\mid K\mid lm\rangle=1. \tag{7.3}$$

That is, from (7.1) and (7.2), and (4.6),

$$\omega_{lm}=(1+B_l)(\omega_L^*+m\omega_c-i(1/\tau_0-1/\tau_l+1/\tau_s)), \tag{7.4}$$

and, in particular

$$\omega_{00}=(1+B_0)(\omega_L^*-i/\tau_s)=\omega_L-i(1+B_0)/\tau_s. \tag{7.5}$$

An interesting point concerning these results is that the collision times appear to be modified by the Landau coefficients. For instance, the mode (0, 0) seems to have a vanishingly small linewidth as $B_0 \to -1$. It will be recalled that the B_l are coefficients of the spin-exchange part of the Landau scattering matrix $\mathcal{B}$ and B_0 is proportional to the number ΔS_z [cf. (2.11) and (2.12)] of correlated spins. If a scattering center tries to flip the spin of an individual electron, it must fight the tendency of the spins to be correlated. So, in effect, it must flip many spins at once. This difficulty increases as the ferromagnetic condition $(B_0 \to -1)$ is approached for then all of the (10^{23}) spins must be flipped. Consequently $(\tau_s)_{\text{effective}} \to \infty$.

The normal modes of the longitudinal magnetization may be treated in the same way as those of the transverse components. From the algorithm (5.12) we find

$$\omega_{lm} = (1+B_l)[m\omega_c - (i/\tau_0) + (i/\tau_l)], \qquad (7.6)$$

which is the same result as that obtained for the transverse case if we set $\omega_L^* = 1/\tau_s = 0$.

Unlike any of the other modes (l, m) the mode (0, 0) of the transverse magnetization M_- (otherwise known as CESR) is broadened only by the spin-lattice time τ_s.

Let us now turn to finite wavelengths and consider the dispersion of these waves:

$$\omega(q) = \omega_{lm} + D(l, m) q^2. \qquad (7.7)$$

The reason for this particular form of dispersion will be apparent presently.

If we take the system (5.11), truncating at some $l = L$ (i.e., $B_l = 0$, $\tau_l = 0$, $l \geq L$), then there will be

$$\sum_{l=0}^{L-1} \sum_{m=-l}^{l} \delta_{ml} = \sum_{l=1}^{L} (2l-1) = L^2$$

equations to solve, in general. For the general matrix element $\langle l'm' | K | lm \rangle$, we use the Bessel-function expansion (see Appendix B) correct to $O((qR_c)^2)$. This gives $J_\mu(X) J_\nu(X) \propto (\frac{1}{2}X)^{\mu+\nu}$, where $\mu = (Y - m_<)$, $\nu = -(Y - m_>)$, $X = qR_c \sin\Delta \sin\theta$. Hence only diagonal elements $(m_> = m_<)$ can be of order unity, since elements for which $m' = m$, $l' \neq l$ will be excluded (to this order) since, in (5.6), we may apply the orthogonality condition (4.13) to prove the point. Then, as can be easily verified, the determinant D is equal to the product of zero-order parts of the diagonal elements plus terms of second order in q arising from multiplying such diagonal contributions with (i) products of pairs of elements for which $\Delta m = m_> - m_< = 1$, (ii) products of pairs of elements for which $\Delta m = 0$, $l' \neq l$, (iii) the second-order contributions from the diagonal elements.

It is now evident why we made the ansatz (7.7). Because the diagonal terms dominate, the system (5.11) is amenable to a perturbation treatment with the small parameter q. It is easier to consider q (rather than qR_c) as the formal parameter because we shall assume that q is small wherever it appears, i.e., $qv_F \ll \omega_c$, ω_L, ω, etc.

Then consider G_{lm} and $\mathcal{G}^{lm}$ as components of their respective vectors G and $\mathcal{G}$. The system of equations (5.11) becomes

$$\mathcal{K} G = \mathcal{G}, \qquad (7.8)$$

where the elements of the matrix $\mathcal{K}$ are [from (7.4)]

$$\langle lm | \mathcal{K} | l'm' \rangle = \beta_l \langle lm | K | l'm' \rangle - \delta_{ll'} \delta_{mm'} \qquad (7.9)$$

and

$$\mathcal{G}^{lm} = \langle 00 | K | lm \rangle / 1 + B_0. \qquad (7.10)$$

We wish to find the eigenvalues $\xi(\alpha)$ belonging to the right and left eigenvectors $\phi(\alpha)$ and $\psi(\alpha)$, respectively, such that, in (7.8)

$$\xi(\alpha) \langle \phi(\alpha), G \rangle = \langle \phi(\alpha), \mathcal{G} \rangle. \qquad (7.11)$$

Introducing the set of basis vectors e_{lm} and their respective dual bases e^{lm}, we can project out G_{00}:

$$\begin{aligned} G_{00} = \langle e^{00}, G \rangle &= \sum_{\alpha} \langle e^{00}, \phi(\alpha) \rangle [1/\xi(\alpha)] \langle \psi(\alpha), \mathcal{G} \rangle \\ &= \sum_{\alpha, l, m} [1/\xi(\alpha)] \phi^{00}(\alpha) \psi_{lm}(\alpha) \langle 00 | K | lm \rangle / (1 + B_0) \end{aligned} \qquad (7.12)$$

by using (7.10) and (7.11).

With q as a small parameter, let

$$\mathcal{K} = \mathcal{K}^{(0)} + \mathcal{K}^{(1)}, \qquad (7.13)$$

where

$$\begin{aligned} \langle lm | \mathcal{K}^{(0)} | lm \rangle &= \beta_l \frac{\omega}{\omega + i/\tau - \omega_L^* - m\omega_c} - 1 \\ &= \xi^{(0)}(l, m). \end{aligned} \qquad (7.14)$$

There is no reason to assume that all the β_l are not all distinct. Then we can use nondegenerate perturbation theory to second order for the perturbed eigenvalues ξ:

$$\xi^{(1)}(l, m) = \langle lm | \mathcal{K}^{(1)} | lm \rangle, \qquad (7.15)$$

$$\xi^{(2)}(l, m) = \sum_{(l'm') \neq (lm)} \frac{\langle lm | \mathcal{K}^{(1)} | l'm' \rangle \langle l'm' | \mathcal{K}^{(1)} lm \rangle}{\xi^{(0)}(lm) - \xi^{(0)}(l'm')}. \qquad (7.16)$$

The first-order eigenfunctions are (omitting order superscripts)

$$\phi^{l'm'}(l, m) = \frac{\langle l'm' | \mathcal{K}^{(1)} | lm \rangle}{\xi^{(0)}(lm) - \xi^{(0)}(l'm')}, \qquad (7.17)$$

$$\psi^{l'm'}(l, m) = \frac{\langle lm | \mathcal{K}^{(1)} | l'm' \rangle}{\xi^{(0)}(lm) - \xi^{(0)}(l'm')}, \qquad (7.18)$$

so that

$$\psi^{l'm'}(lm) = \phi^{l'm'}(lm) (\beta_{l'}/\beta_l), \qquad (7.19)$$

using (7.9).

The evaluation of the eigenvalues and eigenvectors is somewhat tedious. First, we must calculate the three types of matrix elements that contribute to $O(q^2)$. This is shown in Appendix C. Armed with these results it is straightforward to compute, from (7.15) and

(7.16), the perturbed eigenvalues

$$\xi^{(1)}(lm)=(2l+1)(qv_F)^2\sum_{l'=|l-1|,\pm}^{|l+1|}(2l'+1)\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2 \times\left\{\frac{\cos^2\Delta}{(\omega\beta_l)^2}\begin{pmatrix} l & 1 & l' \\ m & 0 & -m\end{pmatrix}^2+\frac{1}{2}\frac{\sin^2\Delta}{(\omega\beta_l)^2-\omega_c^2}\begin{pmatrix} l & 1 & l' \\ m & \pm1 & -(m\pm1)\end{pmatrix}^2\right\}, \quad (7.20)$$

where

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}$$

is the $3-j$ symbol of Wigner[31] (see Appendix C). The quantity $\omega\beta_l$ is to be calculated at $\omega=\omega_{lm}$. The indices $\pm$ in the summation refer to the second term where we add both contributions.

Similarly, we obtain

$$\xi^{(2)}(lm)=(qv_F)^2\left\{\frac{2l+1}{(\omega\beta_l)^2}\sum_{l'}(2l'+1)\frac{\beta_{l'}}{\beta_l-\beta_{l'}}\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l' \\ m & 0 & -m\end{pmatrix}^2\cos^2\Delta +\frac{2l+1}{2\beta_l}\sum\frac{(2l'+1)\beta_{l'}}{(\omega\beta_l\pm\omega_c)[\omega(\beta_l-\beta_{l'})\mp\omega_c]}\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l' \\ m & \pm1 & -(m\pm1)\end{pmatrix}^2\sin^2\Delta\right\}. \quad (7.21)$$

We can now obtain the eigenmodes from the zeroes of the eigenvalues. That is, we solve

$$0=\xi(lm)=\xi^{(0)}(lm)+\xi^{(1)}(lm)+\xi^{(2)}(lm) =\beta_l\frac{\omega_{lm}+D(lm)q^2}{\omega_{lm}+D(lm)q^2+i/\tau-\omega_L{}^*-m\omega_c}-1+\xi^{(1)}+\xi^{(2)}, \quad (7.22)$$

where we have used (7.7) and (7.14). From (7.1) and (7.4) we find that to $O(q^2)$, the dispersion coefficient D is given by

$$D(lm)q^2=(1+B_l)\beta_l\omega_{lm}[\xi^{(1)}(lm)+\xi^{(2)}(lm)]. \quad (7.23)$$

Consequently, using the expressions (7.20) and (7.21) for the perturbations to the eigenvalues we obtain, on substituting these in (7.22), the dispersion relation for the mode (l, m):

$$\omega_{lm}(q)=\omega_{lm}+[D_{\|}(lm)\cos^2\Delta+D_{\perp}(lm)\sin^2\Delta]q^2=0, \quad (7.24)$$

where the component along the magnetic field is

$$D_{\|}(lm)=(2l+1)v_F{}^2\frac{(1+B_l)}{\omega_{lm}}\sum_{l'}\frac{(2l'+1)}{\beta_l-\beta_{l'}}\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l' \\ m & 0 & -m\end{pmatrix}^2 \quad (7.25)$$

and the normal component is

$$D_{\perp}(lm)=\tfrac{1}{2}(2l+1)v_F{}^2(1+B_l)\beta_l\omega_{lm}\sum_{l',\pm}(2l'+1)\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l' \\ m & \pm1 & -(m\pm1)\end{pmatrix}^2 \times\left\{\frac{1}{(\omega_{lm}\beta_l)^2-\omega_c^2}+\frac{\beta_{l'}}{\beta_l}\frac{1}{(\omega_{lm}\beta_l\pm\omega_c)[\omega_{lm}(\beta_l-\beta_{l'})\mp\omega_c]}\right\}. \quad (7.26)$$

From the properties of the $3-j$ symbols (see Appendix C) we see that the q^2 coefficient for the mode (lm) involves the coefficients β_l, β_{l+1}, β_{l-1}. The exception is the (0, 0) mode which depends obviously on β_0, β_1.

These waves were first considered at infinite wavelength by Silin.[4] More recently he has considered the dispersion properties of the $l=0, 1$ modes.[13] We disagree with his results for the $l=1$ cases. The procedure adopted in that work, namely, taking moments of the kinetic equation with spherical harmonics, is valid if it is recognized that for the study of the mode (l, m), the relevant moments are the l and the $(l\pm1)$th. In Ref. 13 only the zeroth and first moments were retained; therefore correct results were found only for the mode (0, 0). Here we have avoided the tricky procedure of taking moments by actually *solving* the kinetic equation (without restriction on q) and of then carrying out the expansion in powers of q^2.

To complete this section we shall calculate the pole strengths for each of the (l, m) modes. For simplicity we shall set $\mathcal{W}=0$ and so shall consider the relative strengths of the modes rather than the linewidths of the associated resonances. For the latter, it is more useful to consider the experimental situation, in which a wave propagates through a finite dielectric (Sec. X).

The pole strength of the mode (l, m) is

$$R_{lm}(q)=\lim_{\omega\to\omega_{lm}(q)}[\omega_{lm}(q)-\omega]\chi(q,\omega)$$
$$=\chi_0^*\lim_{\omega\to\omega_{lm}(q)}[\omega_{lm}(q)-\omega]G_{00}(q,\omega),$$

from (6.3) and (6.4), assuming that the mode derives from a pole of first order. That is, we define the total response

$$G_{00}(q)=\sum_{l,m}R_{lm}(q)/[\omega_{lm}(q)-\omega]$$

and using (7.12) for G_{00}, we have

$$R_{lm}(q)=\chi_0[B_l/(1+B_0)]\omega_{lm}\phi^{00}(lm)\times\sum_{l'm'}\psi_{l'm'}(lm)\langle 00\,|\,K\,|\,l'm'\rangle. \quad (7.27)$$

Thus

$$R_{00}/\chi_0=[B_0/(1+B_0)]\omega_L\langle 00\,|\,K\,|\,00\rangle=\omega_L+O(q^2), \quad (7.28)$$

where all quantities have been evaluated at $\omega=\omega_{00}=\omega_L$. For the higher modes we must include terms of $O(q^2)$:

$$R_{lm}/\chi_0=[B_l/(1+B_0)]\omega_{lm}\phi^{00}(lm)\times\left[\langle 00\,|\,K\,|\,lm\rangle+\frac{B_0(1+B_l)}{B_l(1+B_0)}\phi^{00}(lm)\langle 00\,|\,K\,|\,00\rangle\right].$$

From (7.17) we have

$$\phi^{00}(l,m)=[(1+B_0)B_l^2/(1+B_l)(B_l-B_0)]\langle 00\,|\,K\,|\,lm\rangle,$$

so that, for $l\neq 0$

$$R_{lm}/\chi_0=[B_l^4(1+B_0)/(B_l-B_0)(1+B_l)]\times\omega_{lm}(\langle 00\,|\,K\,|\,lm\rangle)^2. \quad (7.29)$$

It is now straightforward to calculate

$$R_{1,0}/\chi_0=[(1+B_0)(1+B_1)^3/(B_1-B_0)^2\omega_{1,0}]\times\tfrac{1}{3}(qv_F)^2\cos^2\Delta, \quad (7.30)$$

$$R_{1,\pm1}/\chi_0=[(1+B_0)(1+B_1)^3/(B_1-B_0)\omega_{1,\pm1}\pm(1+B_0)\omega_c]\times\tfrac{1}{6}(qv_F)^2\sin^2\Delta. \quad (7.31)$$

Only the (0, 0) mode is of order unity. The higher modes (l, m) have strengths $\propto q^{2l}$ and in the calculation shown here we can calculate correctly only the $l=1$ modes. For $l>1$, $R_{lm}(q)$ is an increasing function of q for small q. Therefore these modes have a maximum oscillator strength for some finite q beyond the range of validity of the expansion used and we are unable to study them without either considerable algebraic effort or a numerical computation. We have adopted the latter alternative.

VIII. SHORT-WAVELENGTH REGIME ($qR_c\gg1$)

In this regime, only the case where the waves propagate normal to the magnetic field ($\Delta=\frac{1}{2}\pi$) do they avoid Landau damping (see Sec. IX). In Appendix D we show that the K-matrix elements are, at most, of order $1/qR_c$. As before we write $G_{00}=N/D$. Then, referring to (5.11) and (5.12), the denominator takes the symbolic form

$$D=\begin{vmatrix}1+\epsilon & \epsilon & \epsilon & \epsilon\\ \epsilon & 1+\epsilon & \epsilon & \epsilon\\ \epsilon & \epsilon & 1+\epsilon & \epsilon\end{vmatrix},$$

where ϵ represents terms of order $(qR_c)^{-1}$ and $(qR_c)^{-3/2}$. Thus, to $O(\epsilon)$, only products of the diagonal terms can contribute so that correct to this order the dispersion relation, corresponding to $D=0$, is

$$1=\sum_{l=0}^{L}\sum_{m=-l}^{l}\beta_l\langle lm\,|\,K\,|\,lm\rangle. \quad (8.1)$$

The details of the calculation of these diagonal matrix elements are contained in Appendix D. We find that, correct in $O(qR_c)^{-3/2}$ [see Eq. (D11)],

$$\langle lm\,|\,K\,|\,lm\rangle=\frac{\omega}{\omega_c}\frac{1}{qR_c}\csc\frac{\pi}{\omega_c}\left(\omega-\omega_L^*-m\omega_c+\frac{i}{\tau}\right)\times\sum_{\lambda=0}^{\infty}\left[P_{\lambda,m}\cos\frac{\pi}{\omega_c}\left(\omega-\omega_L^*-m\omega_c+\frac{i}{\tau}\right)+Q_{\lambda,m}\left(\frac{1}{qR_c}\right)^{-1/2}\sin(2qR_c-\tfrac{1}{4}\pi)\right], \quad (8.2)$$

where $P_{\lambda,m}$ and $Q_{\lambda,m}$ are pure numbers. The term with the former coefficient should dominate as the asymptotic limit $|qR_c|\to\infty$ is approached. We shall foster this by setting

$$\omega_m=\omega_L^*+m\omega_c-i/\tau+\omega_c\eta \quad (8.3)$$

and search for solutions ensuring $|\eta|\ll1$. Perturbing about this trial function we shall require that

$$\langle l'm'\,|\,K\,|\,l'm'\rangle|_{\omega=\omega_m}=\frac{(\omega_L^*+m\omega_c-i/\tau)}{\omega_c}(qR_c)^{-1}(\pi\eta)^{-1}\times\sum_{\lambda=0}^{2l'}\left[P_{\lambda,m'}+\frac{Q_{\lambda,m'}(-1)^{m'-m}}{(qR_c)^{1/2}}\sin(2qR_c-\tfrac{1}{4}\pi)\right], \quad (8.4)$$

to first order in η. Substituting this in (8.1) we see that $\eta\sim|qR_c|^{-1}$, which satisfies our requirement. Then, from (8.3) we have the dispersion relation for short waves:

$$\omega_m(q)=(\omega_L^*+m\omega_c-i/\tau)\left\{1+\sum_{l'm'}\frac{\beta_{l'}}{\pi qR_c}\times\sum_{\lambda=0}^{2l'}\left[P_{\lambda,m}+\frac{Q_{\lambda,m'}(-1)^{m'-m}}{(qR_c)^{1/2}}\sin(2qR_c-\tfrac{1}{4}\pi)\right]\right\}, \quad (8.5)$$

where β_l is to be evaluated at $\omega=\omega_{lm}(0)$. Unlike the long-wave limit [cf. (7.5)], the dependence on the

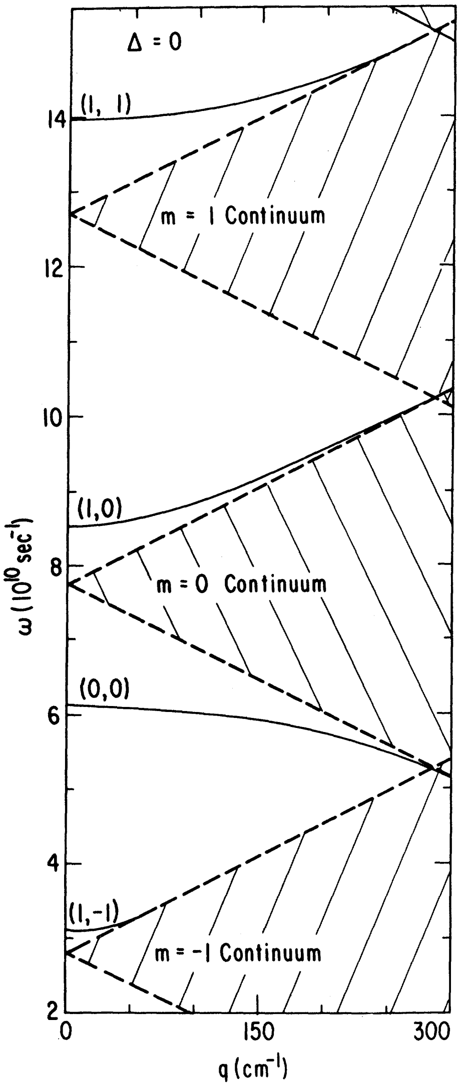

FIG. 2. Excitation spectrum for $\Delta=0$, i.e., propagation along the static magnetic field. We have used values for the parameters appropriate to sodium, except that B_1 has been taken equal to 0.1 in order to improve the visibility of the $l=1$ modes; $\omega_L=6.1264\times10^{10}$ rad/sec. Modes with $|m|>1$ are not shown, and the $l=2$ modes not appear because we have taken $B_2=0$ and they are therefore Landau damped. For this special angle, m is a symmetry quantum number, and the spin waves (00) are not Landau damped by the $m=-1$ continuum. Collisions have been neglected.

Landau parameters has been relegated to the correction term, and so, corresponding to the integer m there is but one mode, instead of $(2l+1)$. That is, as $qR_c\to\infty$ the mode (l, m) which, for $qR_c=0$, began [see (7.5)] as

$$\omega_{lm}=(1+B_l)(\omega_L{}^*+m\omega_c-i/\tau+i/\tau_l)$$

returns to its respective continuum m:

$$\omega_m=(\omega_L{}^*+m\omega_c-i/\tau), \tag{8.6}$$

accompanied by oscillations of diminishing amplitude and increasing frequency.

The scattering rate τ_l does not appear, in order unity, in the short-wave dispersion relation. This is equivalent to saying that in this regime the scatter-in term [cf. (5.11) and (5.12)] is negligible.

Finally we note that in this asymptotic limit the strength of any of the modes (l, m) is also independent of the Landau parameters. This is seen most easily by writing out (5.11) and (5.12) explicitly for G_{00} including an arbitrary number $l=L$ of the B_l. Then, since the matrix elements are at best of order $(qR_c)^{-1}$, including the driving term $\langle 00|K|00\rangle$, which enters only multiplicatively, so all corrections to this term

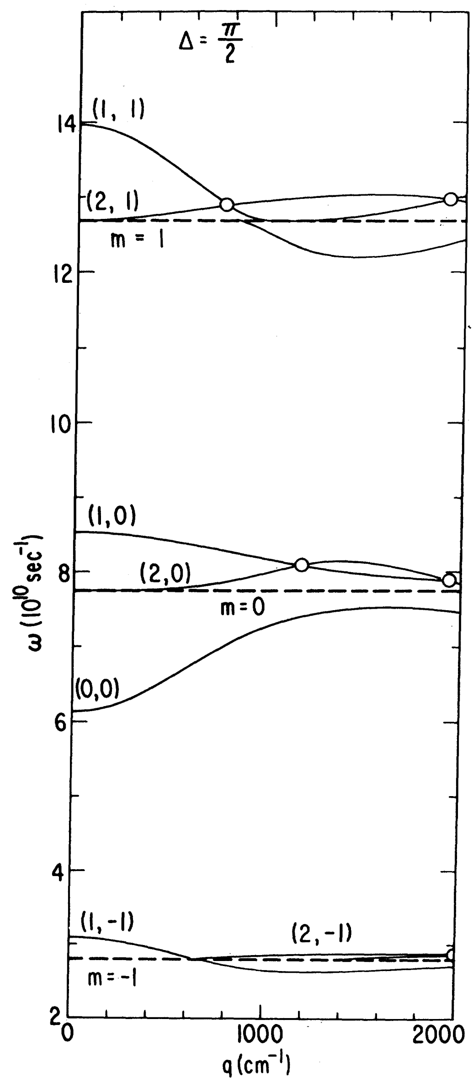

FIG. 3. Excitation spectrum for $\Delta=\frac{1}{2}\pi$, i.e., propagation across the static magnetic field. The parameters have the same values as in Fig. 2. The continua have degenerated to the dotted lines. Note that the scale of q is different from that used in Fig. 2. The $l=2$ modes now appear at finite q, because of the greater range of q values free of single-particle excitations. The frequencies $\omega(q)$ actually oscillate at high q as they approach the continuum, as can be shown by mathematical analysis of the general equations.

will enter as coefficients of terms of order $(qR_c)^{-2}$, and smaller. The Landau coefficients B_l and the collision rates τ_l appear as coefficients of these correction terms only.

As before, the longitudinal modes are amenable to the same procedures and the analogous results readily obtainable by setting $\omega_L^*=0$.

IX. INTERMEDIATE-WAVELENGTH REGIME ($qR_c \approx 1$); NUMERICAL RESULTS

The matrix elements (5.6) possess branch points. These arise from the end-point singularities of the θ integration, that is, at $\theta=0$, π, where the argument of the sine function becomes equal to an integer multiple of π. At these points

$$\omega-\omega_L^* \mp qv_F \cos\Delta + i/\tau = m\omega_c. \tag{9.1}$$

Then, except for the case $\Delta=\frac{1}{2}\pi$ we shall always have a cut in the complex ω plane when investigating the dispersion relation for $\omega(q)$. The cut, of course, arises from the crowding or continuum of poles from each value of θ in $0<\theta<\pi$. We may interpret this continuum as a Landau-damping type of phenomenon. In the presence of the static magnetic field **H**, we can visualize a particle precessing at a frequency $\omega=\omega_L^*+m\omega_c$. The radius of its cyclotron orbit depends on its transverse velocity $v_F \sin\theta$. This describes the motion in the plane normal to **H**. Along it, particles stream with all velocities v_z ranging from 0 up to v_F in either direction. The wave, meanwhile, propagates at an angle Δ with respect to the field and consequently the frequency experienced by a precessing particle will not be ω but rather the Doppler-shifted frequency $\omega - qv_F \cos\Delta \cos\theta$. On adding up the various contributions from the particles on the Fermi surface we find that the interaction between the wave and particles is greatest whenever condition (9.1) is satisfied. We may regard it as the condition for the onset of collective-wave damping or alternatively of the excitation of individual particles.

The intermediate-wavelength regime ($qR_c \approx 1$) is practically inaccessible to analytic investigation. Instead, straightforward numerical methods have been adopted. The results for the spectra of excitations of the transverse susceptibility in the cases $\Delta=0$, $\frac{1}{2}\pi$ appear in Figs. 2 and 3, respectively. Collision effects have been ignored. We use the values $B_0=-0.21$, $B_l=0$, $l \geq 2$ (which are roughly appropriate to sodium) and have set $B_1=0.1$ in order to amplify its effect and to render the $l=1$ modes more visible. The numerical work having been carried out for comparison with experimental data (Schultz and Dunifer[6]), dimensionless variables were not used. The Larmor frequency had the value $\omega_L=6.1264\times10^{10}$ rad/sec, $v_F=0.82\times10^8$ cm/sec, so that $q\approx2000$ cm^{-1} corresponds to $qR_c\approx3.5$ for the frequency range considered ($\omega\approx\omega_L\approx\omega_c$).

Notice that the continua start at $\omega=\omega_L^*+m\omega_c$, $m=0, 1, 2, \cdots$, as described above. Moreover the continua are limited by wedges (locus of the branch points) centered about these starting points with the apex angle 2 arctan(cosΔ) in units $v_F=1$. When we recall [see (7.4)] that the mode (l, m) begins at $\omega=(1+B_l)(\omega_L^*+m\omega_c)$, it is clear that it can remain outside its respective continuum m and hence be a propagating mode only by virtue of its respective Landau parameter being finite.

In the case $\Delta=\frac{1}{2}\pi$ the continua collapse to a line. It was for this reason that we considered only this case in the large qR_c regime. Notice (Fig. 3) the oscillatory behavior of $\omega(q)$ for large q as discussed in Sec. VIII.

A number of curves in Fig. 3 do not appear to be explained in the theory as presented so far. These are the curves that appear to emerge from the various continua m at finite values of q and so would seem to contradict our contention that finite Landau parameters are necessary to avoid the damping of these collective modes. Firstly, these "spurious" modes appear in the case $\Delta=\frac{1}{2}\pi$, where the continua are compressed to a region of measure zero. Secondly, they are nothing but the remnants or "ghosts" of the modes (l, m) described when the respective coefficient B_l is set equal to zero. To see the reason for this we look again at the general dispersion relation (7.24)–(7.26), setting $\Delta=\frac{1}{2}\pi$. For an $l=2$ mode we have $D_\perp(2, m)\neq0$ even if we set $B_2=0$. Similarly, if $B_l=0$ then

$$\omega_{lm}=(\omega_L^*+m\omega_c)\left\{1\mp\tfrac{1}{2}(2l+1)(qv_F)^2\sum_{l',\pm}(2l'+1)\frac{B_{l'}}{1+B_{l'}}\times\left[\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l' \\ m & \pm1 & -(m\pm1)\end{pmatrix}^2\Big/\omega_c\left(\frac{B_{l'}}{1+B_{l'}}(\omega_L^*+m\omega_c)\pm\omega_c\right)\right]\right\}.$$

So, even without the respective B_l the mode (l, m) is present *if* $q\neq0$, that is, but may be visible only in this direction of propagation or close to it. We see this from the condition for visibility at long wavelengths, namely,

$$|\,B_l(\omega_L^*+m\omega_c)+(D_{\parallel}(lm)\cos^2\Delta+D_\perp(lm)\sin^2\Delta)q^2\,|>qv_F\cos\Delta.$$

On account of the greater multiplicity of modes possible in the case $\Delta=\frac{1}{2}\pi$, a certain amount of congestion results in the ω-q plot (Fig. 3). At finite q they appear to cross each other and even to pass through the continuum. In Figs. 4 and 5 we illustrate the effect, neglecting collisions, for clarity. In the top part of Fig. 4 we show, with $\Delta=\frac{1}{2}\pi$, the dispersion of the (1, 1) mode, using $B_0=-0.21$ for three values of B_1. With

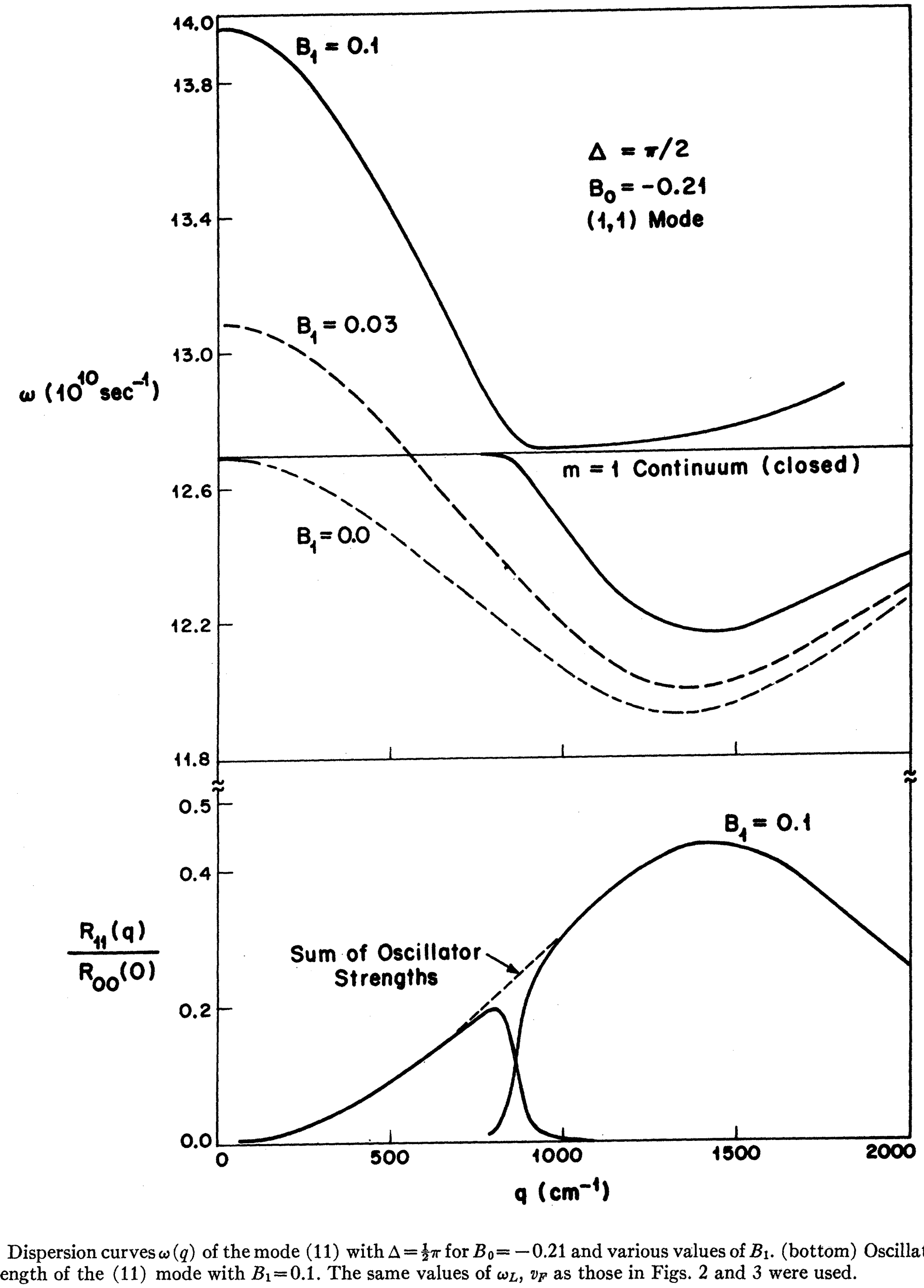

FIG. 4. (top) Dispersion curves $\omega(q)$ of the mode (11) with $\Delta=\frac{1}{2}\pi$ for $B_0=-0.21$ and various values of B_1. (bottom) Oscillator strength of the (11) mode with $B_1=0.1$. The same values of ω_L, v_F as those in Figs. 2 and 3 were used.

$B_1=0$ the mode begins in the $m=1$ continuum line. For $B_1=0.03$ it begins above this line and intersects it at finite q. With $B_1=0.1$ the region of intersection is magnified and the mode splits into two components, each residing on its own side of the continuum. The oscillator strength $R_{11}(q)/R_{00}(0)$ of the mode (with $B_1=0.1$) is shown at the bottom of Fig. 4. While the contribution of either section of the mode to the oscillator strength diminishes rapidly close to the continuum, the total oscillator strength remains a smooth and slowly varying function, which is evidence for the two sections being members of the same "mode."

In Fig. 5 we have opened out the continuum slightly by choosing $\Delta=\frac{1}{2}\pi-0.005$. The (1, 1) mode is shown as before with $B_0=-0.21$, $B_1=0.1$. The mode still appears to cross. In the lower half of the diagram the oscillator strength is seen to behave similarly to that of the preceding case. There is one difference, however.

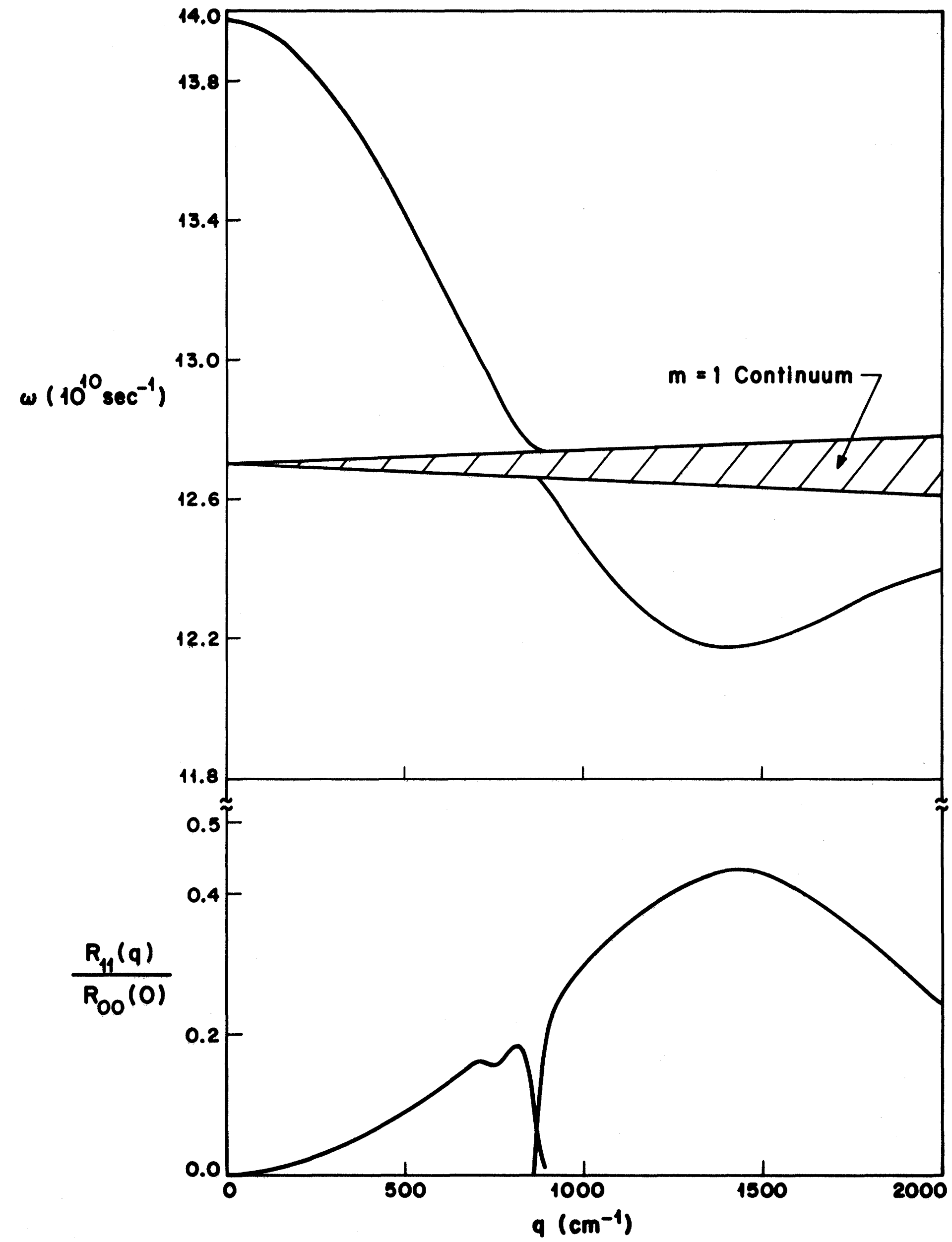

FIG. 5. (top) Dispersion curve $\omega(q)$ of the mode (11) with $\Delta=\frac{1}{2}\pi-0.005$, $B_0=-0.21$, $B_1=0.1$. The continuum has opened up to a narrow fan. (bottom) Oscillator strength of the (11) mode using the same values for B_0 and B_1.

With the slightly opened fan the collective wave can lose energy to individual quasiparticles. The sum of the oscillator strengths is no longer a slowly varying function in the neighborhood of the continuum edge.

X. COMPARISON WITH EXPERIMENT

The problems of transmission of spin waves in a finite, metallic slab, which approximates the experimental conditions of Schultz and Dunifer, has been considered by Lampe and Platzman.[32] They have found, at least approximately, that the ratio of the field transmitted to that incident is

$$\frac{H_{+,-,z}}{H_{\text{inc}}} \propto \sum_{n=0}^{\infty} (-1)^n(2-\delta_{n,0})\chi_{+,-,z}\,(q=n\pi/L), \quad (10.1)$$

where L is the slab thickness. Removing the phase factor $(-1)^n$ gives the result for the field reflected from the slab.

The algorithm described here for calculating χ has been used, in conjunction with the formula above, by Schultz and Dunifer[6] for comparison with their experimental results in sodium. They have retained the

Landau coefficients B_0, B_1, B_2 and the collision times τ_0, τ_s in the computation of χ_- in the neighborhood $\omega \approx \omega_L$. As (10.1) suggests, the eigenvalues of the finite slab are $q_n = n\pi/L$, and resonances are to be expected at frequencies $\omega(q_n)$ corresponding to these eigenvalues. In their paper[6] they show (their Fig. 8) the positions of transmission maxima and minima on a dispersion curve and comparing it with the results of the algorithm they obtain values for the phenomenological parameters B_0, B_1, B_2. In the earlier work of Platzman and Wolff,[22] the expression for χ_- near $\omega = \omega_L$ was given correct to $O(q^2)$. The more general treatment presented here extends the range over which comparison with experiment is possible. Moreover it tests the dependence of the experimental results on B_2 as this parameter does not appear, to $O(q^2)$, in the dispersion equation for the (0, 0) mode. The values reported by Schultz and Dunifer differ from those of their earlier work[5] where comparison with the more approximate theory was made. Using (10.1) they compare (their Fig. 9) the transmitted field spectrum with that obtained from this analysis.

As we may verify [from (7.24)-(7.26)], our form for the long-wavelength dispersion of the (0, 0) mode agrees with the result of Platzman and Wolff provided that (i) we neglect terms of order $1/\omega_L\tau_s$ in the dispersion term and (ii) we replace the collision frequency $1/\tau_{PW}$ used by these authors by the expression $(1+B_1)(1/\tau_0 - 1/\tau_1)$. This latter quantity is of some interest for we know that, rigorously, the resistivity collision time τ_ρ (defined by $\rho = m^*/Ne^2\tau_\rho$) is related simply to our collision times:

$$1/\tau_\rho = 1/\tau_0 - 1/\tau_1.$$

As pointed out in Sec. III our treatment of collisions and of quasiparticle interactions is valid at low temperatures if the coefficients τ_l and B_l are made temperature dependent. Thus, apart from the (near unity) factor $1+B_1$, we expect $\tau_{PW} = \tau_\rho$ at all experimental temperatures, regardless of what the precise temperature dependence of τ_ρ is. This unambiguous prediction does not agree with the measurements of Schultz and Dunifer.[6] It may be, however, that their determination of the dc resistivity ρ suffers from the difficulties (e.g., inhomogeneities) described and remedied by Babiskin and Siebenmann.[33]

XI. COMPARISON OF RESULTS; VISIBILITY OF MODES; CONCLUSIONS

The treatment given here to the analysis of spin waves in an isotropic Fermi liquid is fairly complete in that arbitrary numbers of Landau coefficients have been included as well as the resulting dispersion properties for arbitrary wavelength and direction of propagation. While the work of Silin[3,4] is, of course, preeminent in this field as the source of the original investigations, it is of some interest to compare the results, methods, and limits of generality of subsequent calculations on nonferromagnetic spin waves.

The long-wavelength dispersion coefficients given here [Eqs. (7.24)-(7.26)] reduce to those obtained in[9,11] if we neglect all collisions (Silin[13] differs from these results for reasons given in Sec. VII). The short-wavelength behavior (8.5) is not examined explicitly in these papers nor do the oscillator-strength results (7.28)-(7.31) appear to have been given.

The treatment of collisions given here is more complete than that of[11] the treatment where the spin-relaxation process has not been explicitly included. This is an essential feature in the comparison with experimental results in the CESR. Also, while not a discrepancy of major importance, we note that the collision integral there, although correctly given, is improperly handled in the transport equation, the effect being to include relaxation to true rather than local equilibrium in the "scatter-in" term.

Several features of the present calculation distinguish it from the others mentioned here. Being carried out specifically for comparison with experiment, it contains collision effects consistently throughout. More important, it contains an algorithm for the calculation of the susceptibility including maximum generality. This algorithm is of practical importance,[6] enabling numerical results to be obtained with relative ease. The matrix elements (5.6) have been reduced, in general, to a single quadrative which is an important advantage over other schemes (e.g., Refs. 4 and 11). Thirdly, the derivation of results based on a "molecular-field" model of the Fermi liquid as described elegantly by Nozières[14] is both interesting and useful. Possessing the same information as the transport equation, the integral formulation permits a clearer presentation of the features of the quasiparticle dynamics. The integral expressions for the spin densities or charge densities, (3.18)-(3.20), are in a form directly applicable to boundary- or initial-value problems.[34]

The modes (l, m) analyzed here result from the energy $\frac{1}{2}\gamma \sum \delta H_{lm}^* Y_{lm}$ which a quasiparticle has in the combined Landau and external fields. They would not exist otherwise. Since δH^* varies from point to point, it affects the spatial dispersion of the waves, in addition to changing the spin precession and cyclotron rotation frequencies.

To date, of all of these excitations, only the mode (0, 0) of the transverse magnetization has been observed (Sec. X). Probably the reason for this is the weak spatial dispersion of that experimental situation. We recall, (7.29)-(7.31) that for $qR_c \ll 1$ (also qv_F/ω_L, $qv_F\tau \ll 1$), the oscillator strength R_{lm} decreases rapidly with increasing l. In addition [cf. (7.5)], the linewidth of the mode (l, m) is $(1/\tau - 1/\tau_l + 1/\tau_s)$, and, since $\tau_l \to \infty$ as l increases and as $\tau_0 \ll \tau_s$, so we can see that the $l=0$ mode is the sharpest in definition.

While it would seem difficult to overcome the linewidth problem, it appears more profitable experimentally to investigate the $qR_c \approx 1$ regime, i.e., $L \approx R_c$. In Figs. 4 and 5 we see that the mode (1, 1) is strongest (with $\omega \approx 6.10^{10}$ rad/sec) for $q \approx 1200$–1500 cm^{-1} or $L \approx 2.10^{-3}$ cm for the fundamental geometric harmonic ($m=1$). Higher frequencies, while improving $\omega\tau$ values would demand thinner samples.

With the advent of uv lasers, light sources are becoming available that can penetrate metals, i.e., $\omega >$ plasma frequency. As suggested by Genkin and Genkin,[35] the use of Raman scattering could be a valuable tool for investigating the spin waves. Under such experimental conditions the wavelength $\lambda = 2\pi/q$ of the scattered radiation is fixed and the observed quantity is the frequency shift $\Delta\omega$ of the natural frequency excited. For a line of wavelength $\lambda \approx 3000$ Å, $qv_F \approx 10^{13}$ sec^{-1} in sodium, so that $qv_F\tau \gg 1$. However, even with fields $H \approx 50$ kOe, we have $qR_c \approx 20$, which is in the region of strong dispersion. As these authors have remarked this region has not been investigated so far. Unfortunately, the only Landau parameter to appear [to zero order in $(qR_c)^{-1}$] is B_0, as we see from (8.5), the size of the effect for the other parameters being $\approx B_l/60 \lesssim 1\%$ for the situation just described. High resolution would be required to observe the effect.

Although the analysis presented here includes the effects of (linear) interaction of the various modes we have not discussed how they might interact with other normal modes of the system, e.g., the cyclotron modes. We should have to solve Maxwell's equations[36]

$$\nabla \cdot \mathbf{E} = 4\pi[-2e/(2\pi)^3] \int d\mathbf{k}\, \delta n(\mathbf{k}) + 4\pi \text{ (positive background charge)}, \tag{11.1}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{11.2}$$

$$\nabla \times \mathbf{E} + c^{-1}(\partial \mathbf{H}/\partial t) = 0, \tag{11.3}$$

$$\nabla \times \mathbf{H} - c^{-1}(\partial \mathbf{E}/\partial t) = (4\pi/c)(\mathbf{J}_P + \mathbf{J}_M), \tag{11.4}$$

where

$$\mathbf{J}_P = -[2e/(2\pi)^3] \int d\mathbf{k}\, \delta\bar{n}(\mathbf{k})(\mathbf{k}/m^*), \tag{11.5}$$

and

$$\mathbf{J}_M = c\nabla \times \mathbf{M} \tag{11.6}$$

is the magnetization current with $\mathbf{M}$ given in (6.1)–(6.4). The polarization current is of order

$$J_P \approx (\mathfrak{N}_F e v_F \delta E_q/\omega) e v_F,$$

where $\delta E_q(\omega)$ is the rf electric field. Similarly, from (11.4), (11.6), and (6.4), $J_M \approx cq\chi\delta H_q(\omega) \approx cq\gamma^2 \mathfrak{N}_F \delta H_q$. Consequently,

$$\frac{J_P(q,\omega)}{J_M(q,\omega)} \approx \frac{ev_F \delta E_q/\omega}{\gamma \delta H_q} \frac{v_F}{\hbar q/m}, \tag{11.7}$$

where we have reinserted $\hbar$ into the definition of the magnetic moment $\gamma = (e\hbar/mc)(g/2)$. This fraction is proportional to the ratio of the energy gained by a quasiparticle from the electric field to that of its moment in the magnetic field. From (11.3) we have $cq\delta E_q \approx \omega \delta H_q$, so that in (11.7) we find

$$\frac{J_P(q,\omega)}{J_M(q,\omega)} \approx \left(\frac{v_F}{\hbar q/m}\right)^2. \tag{11.8}$$

Under the experimental conditions described in Sec. X, $q \approx 10^3$ cm^{-1}, so that for $v_F \approx 10^8$ cm sec^{-1}, the ratio is of order 10^{10}. The effects of magnetization are very small. However, we have assumed the medium to be unbounded. Because of the skin effect in a metal, $\delta E \ll \delta H$ beyond the skin layer, so that a more careful treatment is necessary. Also we have not included the effects of a static magnetic field on the magnitude of J_P, J_M. The analysis required to study in detail the interaction between the magnetization and conduction modes will be given elsewhere.[34] Suffice it to say that for the alkali metals (which are weakly magnetic) the treatment used here remains valid.

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APPENDIX A

Starting with the matrix element (5.4) expressed as

$$\langle l'm' | K(X,Y) | lm \rangle = -\frac{i\omega}{\omega_c} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y_{lm}^*(\theta,\phi) \int_0^{\infty} d\Phi\, e^{iY\Phi} \times \exp i\{X(\theta)[\sin(\phi-\Phi) - \sin\phi]\} Y_{l'm'}(\theta, \phi-\Phi),$$

we use the property[37] of Bessel functions

$$\exp(iu \sin\phi) = \sum_{n=-\infty}^{\infty} J_n(u) e^{in\phi}$$

and the definition $C_{lm}=Y_{lm}e^{-im\phi}$ to perform the ϕ integration:

$$\langle l'm' \mid K \mid lm\rangle = -\frac{2\pi i\omega}{\omega_c}\sum_{n,n'}\int_0^\pi d\theta \sin\theta\, C_{lm}(\theta)C_{l'm'}J_n(X(\theta))J_{n'}\delta_{-m+n-n'+m'}\int_0^\infty d\Phi \exp\{i[Y(\theta)-n-m']\Phi\}$$

$$= \frac{2\pi\omega}{\omega_c}\sum_n\int_0^\pi d\theta \sin\theta\, C_{lm}C_{l'm'}\frac{J_n(X(\theta))J_{n+m'-m}}{Y(\theta)-(n+m')}. \tag{A1}$$

The summation of the Bessel functions may now be performed as follows. Recall the addition theorem[37]

$$\sum_n J_n(X)J_{n+p}(X)e^{in\phi}=e^{ip\psi}J_p(X'),$$

where

$$X'=2X\sin\tfrac{1}{2}\phi,\qquad \psi=\tfrac{1}{2}(\pi-\phi).$$

Integrate on the left- and right-hand sides of the identity with

$$\int_0^{2\pi} d\phi\, e^{-i\eta\phi},$$

setting $p=m'-m$. On the left-hand side we have

$$\sum_n\int_0^{2\pi} d\phi\, J_n(X)J_{n+m'-m}(X)\exp[-i(\eta-n)\phi]=\sum_n\frac{J_n(X)J_{n+m'-m}(X)}{i(\eta-n)}[1-\exp(-i2\pi\eta)]. \tag{A2}$$

On the right-hand side we have

$$\int_0^{2\pi} d\phi \exp[i(m'-m)(\pi-\phi)/2]J_{m'-m}(2X\sin\tfrac{1}{2}\phi)e^{-i\eta\phi}$$

$$=2\exp[(i\pi/2)(m'-m)]\int_0^\pi dt\, J_{m'-m}(2X\sin t)\exp[-i(m'-m+2\eta)t] \tag{A3}$$

$$=2\exp[-(i\pi/2)(m'-m)]\int_0^\pi dt\, J_{-(m'-m)}(2X\sin t)\exp[-i(m'-m+2\eta)t], \tag{A4}$$

where, in the last line, we have used

$$J_n(Z)=e^{i\pi n}J_{-n}(Z). \tag{A5}$$

Next, we apply the result[38]

$$\int_0^\pi dt\, J_{2\nu}(2X\sin t)e^{i2\beta t}=\pi e^{i\pi\beta}J_{\nu-\beta}(X)J_{\nu+\beta},\qquad \mathrm{Re}\nu>-\tfrac{1}{2}$$

to the former, (A3), of the two integrals above if $m'\geqslant m$ and to the latter (A4) if $m\geqslant m'$. Equating the respective results to the infinite sum (A2) we find

$$\sum_n\frac{J_nJ_{n+m'-m}}{\eta-n}=\frac{\pi}{\sin\pi\eta}J_{\eta+m'-m}J_{-\eta},\qquad m'\geqslant m$$

$$=(-1)^{m'-m}\frac{\pi}{\sin\pi\eta}J_\eta J_{-\eta+m-m'},\qquad m\geqslant m'.$$

By setting $\eta=Y-m'$ we obtain the general result

$$\sum_n\frac{J_n(X)J_{n+m'-m}(X)}{Y-(n+m')}=(-1)^{m_>}\frac{\pi}{\sin\pi Y}J_{Y-m_<}(X)J_{-(Y-m_>)}(X), \tag{A6}$$

where $m_>$ is the greater and $m_<$ the lesser of (m, m'). Inserting this conclusion in (A1), the K-matrix element is reduced to a single quadrature:

$$\langle l'm' \mid K \mid lm\rangle=(-1)^{m_>}\frac{2\pi^2\omega}{\omega_c}\int_0^\pi d\theta\sin\theta\, C_{lm}(\theta)C_{l'm'}(\theta)\frac{J_{Y-m_<}(X(\theta))J_{-(Y-m_>)}(X(\theta))}{\sin\pi Y(\theta)}. \tag{A7}$$

This last integral can also be performed[39] if either X or Y is independent of θ but, for our purposes, it is more convenient in the present form.

APPENDIX B

From the power series representation of a Bessel function

$$J_\mu(X)=(\tfrac{1}{2}X)^\mu \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(\mu+n+1)} (\tfrac{1}{2}X)^{2n},$$

it can be shown that the product of two Bessel functions may be written

$$J_\mu(X)J_\nu(X)=(\tfrac{1}{2}X)^{\mu+\nu} \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\mu+\nu+2n+1)}{\Gamma(\mu+\nu+n+1)\Gamma(n+1)\Gamma(\mu+n+1)\Gamma(\nu+n+1)} (\tfrac{1}{2}X)^{2n}.$$

It is a property of the gamma function Γ that

$$\Gamma(Y)\Gamma(1-Y)=\pi/\sin\pi Y \qquad (Y \text{ not integral}).$$

Then we can expand the series above:

$$\frac{J_{Y-m_<}J_{-(Y-m_>)}}{\sin\pi Y}=\pi^{-1}(\tfrac{1}{2}X)^{m_>-m_<}\left\{\frac{\Gamma(Y)}{\Gamma(Y-m_<+1)}\frac{\Gamma(1-Y)}{\Gamma(1-Y+m_>)}-\frac{(m_>-m_<+2)\Gamma(Y)\Gamma(1-Y)}{(Y-m_<+2)\Gamma(2-Y+m_>)}(\tfrac{1}{2}X)^2+\cdots\right\}.$$

For $X=0$ it is clear that

$$\begin{aligned} J_{Y-m_<}J_{-(Y-m_>)}/\sin\pi Y &= 0 && \text{for } m_>\neq m_< \\ &= (-1)^m/(Y-m) && \text{for } m_>=m_<=m. \end{aligned}$$

Finally, for calculation of the matrix elements $\langle lm \mid K \mid l'm'\rangle$ we need

$\Delta m=0$,

$$[\pi J_Y(X)J_{-Y}(X)/\sin\pi Y]=Y^{-1}\{1-\tfrac{1}{2}[X^2/(1-Y^2)]+O(X^4)\};$$

$\Delta m=1$,

$$[\pi J_{\pm Y}(X)J_{\mp(Y\mp1)}(X)/\sin\pi Y]=\tfrac{1}{2}X[Y(1\mp Y)]^{-1}+O(X^3).$$

APPENDIX C

The three types of K-matrix elements contributing to the small-q analysis may be calculated as follows from the properties given in the two preceding appendixes.

(i) $\Delta m=\pm1$

From (A1) we can show that

$$\langle l', m\pm1 \mid K \mid l, m\rangle=\pi\omega(qv_F\sin\Delta)\int_0^\pi d\theta\,\sin^2\theta\frac{C_{l',m\pm1}C_{l,m}}{[\bar{\omega}-m\omega_c-qv_F\cos\Delta\cos\theta][\bar{\omega}-(m\pm1)\omega_c-qv_F\cos\Delta\cos\theta]}, \quad \text{(C1)}$$

where

$$\bar{\omega}=\omega+i/\tau-\omega_L^*. \quad \text{(C1)}$$

We shall make use of the Wigner $3-j$ symbols[31]:

$$\int_{\text{unit sphere}} d^2\hat{k}\,Y_{l_1m_1}Y_{l_2m_2}Y_{l_3m_3}=\left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right)^{1/2}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}, \quad \text{(C2)}$$

where the integral vanishes unless

$$m_1+m_2+m_3=0, \quad \text{(C2')}$$

$$|l_1-l_2|\leqslant l_3\leqslant|l_1+l_2|, \quad \text{(C2'')}$$

and in the case where $m_1=m_2=m_3=0$,

$$(l_1+l_2+l_3) \quad \text{is even.} \quad \text{(C2''')}$$

Recalling the definition (5.7) of $C_{lm}(\theta)$, it is straightforward to obtain

$$\langle l', m\pm1 \mid K \mid l, m\rangle=-\frac{\omega}{\sqrt{2}}\frac{qv_F\sin\Delta}{(\bar{\omega}-m\omega_c)(\bar{\omega}-(m\pm1)\omega_c)}$$

$$\times((2l+1)(2l'+1))^{1/2}\begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & 1 & l' \\ m & \pm1 & -(m\pm1)\end{pmatrix}+O(q^3). \quad \text{(C3)}$$

(ii) $\Delta l\neq 0$, $\Delta m=0$

Proceeding as in case (i), we have

$$\langle l'm \mid K \mid lm\rangle = \frac{2\pi\omega}{\bar{\omega}-m\omega_c}\int_0^\pi d\theta\,\sin\theta\,C_{lm}C_{l'm}\left(1+\frac{qv_F\cos\Delta\cos\theta}{\bar{\omega}-m\omega_c}\right)+O(q^3)$$

$$=\frac{\omega}{\bar{\omega}-m\omega_c}\left\{((2l+1)(2l'+1))^{1/2}\begin{pmatrix} l' & 0 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 0 & l\\ m & 0 & -m\end{pmatrix}+\frac{qv_F\cos\Delta}{\bar{\omega}-m\omega_c}((2l+1)(2l'+1))^{1/2}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ m & 0 & -m\end{pmatrix}\right\}$$

$$=\frac{\omega}{(\bar{\omega}-m\omega_c)^2}((2l+1)(2l'+1))^{1/2}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ m & 0 & -m\end{pmatrix}qv_F\cos\Delta+O(q^3), \quad \text{(C4)}$$

where we have used (C2″) and the fact $l'\neq l$ to eliminate the former of the two pairs of $3-j$ symbols.

(iii) $\Delta m=0$, $\Delta l=0$

$$\langle lm \mid K \mid lm\rangle = (-1)^m 2\pi^2\omega\int_0^\pi \frac{d\theta\,\sin\theta\,C_{lm}^2}{\bar{\omega}-m\omega_c-qv_F\cos\Delta\cos\theta}\left[1-\frac{1}{2}\frac{(qv_F\sin\Delta\sin\theta)^2}{\omega_c^2-(\bar{\omega}-m\omega_c)^2}\right]+O(q^4)$$

$$=\frac{\omega}{\bar{\omega}-m\omega_c}\left\{1+\int_0^\pi d\theta\,\sin\theta\,C_{lm}^2\left[\frac{(qv_F\cos\Delta\cos\theta)^2}{(\bar{\omega}-m\omega_c)^2}-\frac{1}{2}\frac{(qv_F\sin\Delta\sin\theta)^2}{\omega_c^2-(\bar{\omega}-m\omega_c)^2}\right]\right\}+O(q^4), \quad \text{(C5)}$$

where we have expanded the denominator correct to $O(q^2)$ and found that the term of $O(q)$ vanished from symmetry considerations. We now use the decomposition property of spherical harmonics,

$$Y_{l_1m_1}(\hat{k})Y_{l_2m_2}(\hat{k})=\sum_{\lambda=|l_1-l_2|}^{|l_1+l_2|}\sum_{\mu=-\lambda}^{\lambda}(-1)^\mu\left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right)^{1/2}\begin{pmatrix} l_1 & l_2 & \lambda\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & \lambda\\ m_1 & m_2 & \mu\end{pmatrix}Y_{\lambda,-\mu}(\hat{k}), \quad \text{(C6)}$$

to show, for instance, that

$$\int_0^\pi d\theta\,\sin\theta\,C_{lm}^2\cos^2\theta=\frac{(-1)^m}{2\pi}\frac{4\pi}{3}\int d^2\hat{k}\;Y_{lm}Y_{l,-m}Y_{1,0}^2$$

$$=(2\pi)^{-1}\sum_{l'=|l-1|}^{|l+1|}(2l+1)(2l'+1)\begin{pmatrix} l & 1 & l'\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l'\\ m & 0 & -m\end{pmatrix}^2,$$

where we have used

$$Y_{lm}^*=(-1)^m Y_{l,-m} \quad \text{(C7)}$$

and

$$Y_{1,0}=(3/4\pi)^{1/2}\cos\theta. \quad \text{(C8)}$$

We shall also need

$$Y_{1,\pm 1}=\mp(3/8\pi)^{1/2}\sin\theta e^{\pm i\phi}. \quad \text{(C9)}$$

Finally, we obtain

$$\langle lm \mid K \mid lm\rangle = \frac{\omega}{\bar{\omega}-m\omega_c}\left\{1+\left(\frac{qv_F\cos\Delta}{\bar{\omega}-m\omega_c}\right)^2\sum_{l'=|l-1|}^{|l+1|}(2l+1)(2l'+1)\begin{pmatrix} l & 1 & l'\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l'\\ m & 0 & -m\end{pmatrix}^2+\frac{1}{2}\frac{(qv_F\sin\Delta)^2}{(\bar{\omega}-m\omega_c)^2-\omega_c^2}\sum_{l'=|l-1|,\pm}^{|l+1|}(2l+1)(2l'+1)\begin{pmatrix} l & 1 & l'\\ 0 & 0 & 0\end{pmatrix}^2\begin{pmatrix} l & 1 & l'\\ m & \pm 1 & -(m\pm 1)\end{pmatrix}^2\right\}. \quad \text{(C10)}$$

The spherical harmonics Y_{lm} used here are those of Condon and Shortley.[29] They are defined [cf. (5.7)]

$$Y_{lm}(\theta\phi)=(-1)^{(|m|-m)/2}\left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}\sin^{|m|}\theta\left(\frac{d}{d\cos\theta}\right)^m P_l(\cos\theta)\equiv C_{lm}e^{im\phi}, \quad \text{(C11)}$$

such that the C_{lm} are real. P_l is a Legendre polynomial.

APPENDIX D

We shall derive here expressions for the K-matrix elements for the case $\Delta=\frac{1}{2}\pi$ in the asymptotic limit $|qR_c|\gg 1$ or $|(\omega-\omega_L^*-m\omega_c+i/\tau)/\omega_c|$. We have, from (5.8) and (5.9),

$$X=(qv_F/\omega_c)\sin\theta,\qquad Y=(\omega-\omega_L^*+i/\tau)/\omega_c. \tag{D1}$$

The Bessel functions $J_Y(X)$ may be replaced by their asymptotic form

$$J_Y(X)=(2/\pi X)^{1/2}\cos(X-\tfrac{1}{2}\pi Y-\tfrac{1}{4}\pi)+O(X^{-3/2}), \tag{D2}$$

so that correct to $O(X^{-3/2})$, a typical matrix element is

$$\langle l'm'\mid K\mid lm\rangle=(2\pi\omega/\omega_c)(-1)^{m_>}(\omega_c/qv_F)\csc\pi Y$$
$$\times\int_0^\pi d\theta\, C_{lm}(\theta)C_{l'm'}(\theta)[\cos\pi(Y-m_>+\tfrac{1}{2}(m_>-m_<))+\sin(2X-\tfrac{1}{2}\pi(m_>-m_<))], \tag{D3}$$

which is clearly of order $(qR_c)^{-1}$, at most. We shall now concentrate on the diagonal elements $(l=l',\ m_>=m_<=m)$.

In Appendix C we found

$$C_{lm}^2=(-1)^m Y_{lm}Y_{l,-m}$$
$$=(-1)^m(2l+1)\sum_{\lambda=0}^{2l}\left(\frac{2\lambda+1}{4\pi}\right)^{1/2}\begin{pmatrix} l & l & \lambda\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l & \lambda\\ m & -m & 0\end{pmatrix}Y_{\lambda 0}, \tag{D4}$$

and, from (C2‴), λ is an even integer. But

$$Y_{\lambda 0}(\theta)=C_{\lambda 0}=[(2\lambda+1)/4\pi]^{1/2}P_\lambda(\cos\theta) \tag{D5}$$

from (C11). Now

$$\int_0^\pi d\theta\, P_\lambda(\cos\theta)=\int_{-1}^{1}\frac{du}{(1-u^2)^{1/2}}P_\lambda(u)\qquad(\lambda\text{ even})$$
$$=[\Gamma(\tfrac{1}{2}(\lambda+1))/\Gamma(\tfrac{1}{2}\lambda+1)]^2, \tag{D6}$$

according to Gradshteyn and Ryzhik.[40] This deals with the first of the integrals in (D3). For the second we note that since $P_\lambda(\cos\theta)$ is even about $\theta=\frac{1}{2}\pi$ as λ is an even integer, so the integral is nonzero. We take the series representation

$$P_\lambda(\cos\theta)=\sum_{r=0}^{\lambda/2}\frac{(-1)^r(2\lambda-2r)!}{2^\lambda r!(\lambda-r)!(\lambda-2r)!}(\cos\theta)^{\lambda-2r}, \tag{D7}$$

which indicates that in (D3) we require the integral

$$\int_0^\pi d\theta\,\cos^{\lambda-2r}\theta\,\sin(2Z\sin\theta),\qquad |Z|\gg 1.$$

In the range of integration the outer sine function varies rapidly except close to $\theta=\frac{1}{2}\pi$ which, as can be verified, is a point of stationary phase. Let $\theta=\frac{1}{2}\pi+\psi$ so that $\cos\theta\sim\psi+O(\psi^3)$. Then the integral above may be replaced by

$$\int d\psi\,\psi^{\lambda-2r}\sin[2Z(1-\tfrac{1}{2}\psi^2)]\sim(2i)^{-1}\int_{-\infty}^{\infty}d\psi\,\psi^{\lambda-2r}[\exp(i2Z)\exp(-iZ\psi^2)-\exp(-i2Z)\exp(iZ\psi^2)]. \tag{D8}$$

From the Gaussian integral

$$I_0=\int_{-\infty}^{\infty}d\psi\,\exp(-a\psi^2)=\left(\frac{\pi}{a}\right)^{1/2}, \tag{D9}$$

we can easily show that

$$I_n\sim 1/a^{n+1/2}.$$

Since we do not require terms of order smaller than $Z^{-1/2}$ in the integrals in (D3), we may discard all terms from the expansion (D7) except for $\lambda=2r$. But from (D8) and (D9),

$$\int_0^\pi d\theta\,\sin(2Z\sin\theta)=\left(\frac{\pi}{Z}\right)^{1/2}\sin(2Z-\tfrac{1}{4}\pi)+O(Z^{-3/2}). \tag{D10}$$

This deals with the second of the integrals of (D3). Assembling our results we have, correct to $O(\omega/qv_F)^{3/2}$

$$\langle lm \mid K \mid lm\rangle = (-1)^m \frac{\omega}{2qv_F} (2l+1) \sum_{\lambda=0}^{2l} (2\lambda+1) \begin{pmatrix} l & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & \lambda \\ m & -m & 0 \end{pmatrix} \csc\pi(Y-m) \times \left\{ \frac{\Gamma^2(\frac{1}{2}(\lambda+1))}{\Gamma^2(\frac{1}{2}\lambda+1)} \cos\pi(Y-m) + \frac{(-1)^{\lambda/2}\lambda!}{2^\lambda(\frac{1}{2}\lambda)!^2} \left(\frac{\pi\omega_c}{qv_F}\right)^{1/2} \sin\left(\frac{2qv_F}{\omega_c} - \frac{1}{4}\pi\right) \right\}. \quad \text{(D11)}$$

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